Application of Random Matrix Theory to Multivariate Statistics

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Abstract

This is an expository account of the edge eigenvalue distributions in random matrix theory and their application in multivariate statistics. The emphasis is on the Painlevé representations of these distribution functions.

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1 Multivariate Statistics

1.1 Wishart distribution

The basic problem in statistics is testing the agreement between actual observations and an underlying probability model. Pearson in 1900 [27] introduced the famous χ^2 *test* where the sampling distribution approaches, as the sample size increases, to the χ^2 distribution. Recall that if X_j are independent and identically distributed standard normal random variables, $N(0,1)$, then the distribution of

$$\chi_n^2 := X_1^2 + \cdots + X_n^2 \tag{1.1}$$

has density

$$f_n(x) = \begin{cases} \frac{1}{2^{n/2}\,\Gamma(n/2)}\, x^{n/2-1} e^{-x/2} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases} \tag{1.2}$$

where $\Gamma(x)$ is the gamma function.

In classical *multivariate statistics*[1] it is commonly assumed that the underlying distribution is the multivariate normal distribution. If X is a $p \times 1$-variate normal with $\mathbb{E}(X) = \mu$ and $p \times p$ covariance matrix $\Sigma = \text{cov}(X) := \mathbb{E}\left((X - \mu) \otimes (X - \mu)\right),$[2] denoted $N_p(\mu, \Sigma)$, then if $\Sigma > 0$ the density function of X is

$$f_X(x) = (2\pi)^{-p/2} \left(\det \Sigma\right)^{-1/2} \exp\left[-\frac{1}{2}\left(x - \mu, \Sigma^{-1}(x - \mu)\right)\right], \ x \in \mathbb{R}^p,$$

where $(\cdot, \cdot)$ is the standard inner product on $\mathbb{R}^p$.

It is convenient to introduce a matrix notation: If X is a $n \times p$ matrix (the *data matrix*) whose rows X_j are independent $N_p(\mu, \Sigma)$ random variables,

$$X = \begin{pmatrix} \longleftarrow & X_1 & \longrightarrow \\ \longleftarrow & X_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & X_n & \longrightarrow \end{pmatrix},$$

then we say X is $N_p(\mathbf{1} \otimes \mu, I_n \otimes \Sigma)$ where $\mathbf{1} = (1, 1, \ldots, 1)$ and I_n is the $n \times n$ identity matrix. We now introduce the multivariate generalization of (1.1).

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[1]There are many excellent textbooks on multivariate statistics, we mention Anderson [1], Muirhead [26], and for a shorter introduction, Bilodeau and Brenner [4].

[2]If u and v are vectors we denote by $u \otimes v$ the matrix with (i, j) matrix element $u_i v_j$.

Definition 1.1. *If $A = X^t X$, where the $n \times p$ matrix X is $N_p(0, I_n \otimes \Sigma)$, then A is said to have Wishart distribution with n degrees of freedom and covariance matrix Σ. We write A is $W_p(n, \Sigma)$.*

To state the generalization of (1.2) we first introduce the *multivariate Gamma function*. If $\mathcal{S}_m^+$ is the space of $p \times p$ positive definite, symmetric matrices, then

$$\Gamma_p(a) := \int_{\mathcal{S}_p^+} e^{-\operatorname{tr}(A)} \left(\det A\right)^{a-(p+1)/2} dA$$

where $\operatorname{Re}(a) > (m-1)/2$ and dA is the product Lebesgue measure of the $\frac{1}{2}p(p+1)$ distinct elements of A. By introducing the matrix factorization $A = T^t T$ where T is upper-triangular with positive diagonal elements, one can evaluate this integral in terms of ordinary gamma functions, see, page 62 in [26]. Note that $\Gamma_1(a)$ is the usual gamma function $\Gamma(a)$. The basic fact about the Wishart distributions is

Theorem 1.2 (Wishart [38]). *If A is $W_p(n, \Sigma)$ with $n \geq p$, then the density function of A is*

$$\frac{1}{2^{p\,n/2}\,\Gamma_p(n/2)\,\left(\det \Sigma\right)^{n/2}}\, e^{-\frac{1}{2}\operatorname{tr}(\Sigma^{-1}A)}\, \left(\det A\right)^{(n-p-1)/2}. \tag{1.3}$$

For $p = 1$ and $\Sigma = 1$ (1.3) reduces to (1.2). The case $p = 2$ was obtained by Fisher in 1915 and for general p by Wishart in 1928 using geometrical arguments. Most modern proofs follow James [20]. The importance of the Wishart distribution lies in the fact that the *sample covariance matrix*, S, is $W_p(n, \frac{1}{n}\Sigma)$ where

$$S := \frac{1}{n}\sum_{j=1}^{N}\left(X_i - \overline{X}\right) \otimes \left(X_j - \overline{X}\right),\ N = n + 1,$$

and X_j, $j = 1, \ldots, N$, are independent $N_p(\mu, \Sigma)$ random vectors, and $\overline{X} = \frac{1}{N}\sum_j X_j$.

Principle component analysis,[3] a multivariate data reduction technique, requires the eigenvalues of the sample covariance matrix; in particular, the

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[3]See, for example, Chap. 9 in [26], and [22] for a discussion of some current issues in principle component analysis.

largest eigenvalue (the largest principle component variance) is most important. The next major result gives the joint density for the eigenvalues of a Wishart matrix.

Theorem 1.3 (James [21]). *If A is $W_p(n, \Sigma)$ with $n \geq p$ the joint density function of the eigenvalues $\ell_1, \ldots, \ell_p$ of A is*

$$\frac{\pi^{p^2/2} 2^{-pn/2} \left(\det \Sigma\right)^{-n/2}}{\Gamma_p(p/2)\Gamma_p(n/2)} \prod_{j=1}^{p} \ell_j^{(n-p-1)/2} \prod_{j<k} (\ell_j - \ell_k)$$
$$\cdot \int_{\mathbb{O}(p)} e^{-\frac{1}{2}\operatorname{tr}(\Sigma^{-1} HLH^t)} \, dH \qquad (1.4)$$

where $\mathbb{O}(p)$ is the orthogonal group of $p \times p$ matrices, dH is normalized Haar measure and L is the diagonal matrix $\operatorname{diag}(\ell_1, \ldots, \ell_p)$. (We take $\ell_1 > \ell_2 > \cdots > \ell_p$.)

Remark 1.4. The difficult part of this density function is the integral over the orthogonal group $\mathbb{O}(p)$. There is no known closed formula for this integral though James and Constantine (see Chap. 7 in [26] for references) developed the theory of *zonal polynomials* which allow one to write infinite series expansions for this integral. However, these expansions converge slowly; and zonal polynomials themselves, lack explicit formulas such as are available for Schur polynomials. For complex Wishart matrices, the group integral is over the unitary group $\mathbb{U}(p)$; and this integral can be evaluated using the Harish-Chandra-Itzykson-Zuber integral [39].

There is one important case where the integral can be (trivially) evaluated.

Corollary 1.5. *If $\Sigma = I_p$, then the joint density (1.4) simplifies to*

$$\frac{\pi^{p^2/2} 2^{-pn/2} \left(\det \Sigma\right)^{-n/2}}{\Gamma_p(p/2)\Gamma_p(n/2)} \prod_{j=1}^{p} \ell_j^{(n-p-1)/2} \exp\left(-\frac{1}{2}\sum_j \ell_j\right) \prod_{j<k} (\ell_j - \ell_k). \quad (1.5)$$

1.2 An example with $\Sigma \neq cI_p$

This section uses the theory of zonal polynomials as can be found in Chap. 7 of Muirhead [26] or Macdonald [23]. This section is not used in the remainder of the chapter. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition into not more than p

parts. We let $C_\lambda(Y)$ denote the zonal polynomial of Y corresponding to λ. It is a symmetric, homogeneous polynomial of degree $|\lambda|$ in the eigenvalues $y_1, \ldots, y_p$ of Y. The normalization we adopt is defined by

$$(\operatorname{tr} Y)^k = (y_1 + \cdots + y_p)^k = \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq p}} C_\lambda(Y).$$

The fundamental integral formula for zonal polynomials is[4]

Theorem 1.6. *Let $X, Y \in \mathcal{S}_p^+$, then*

$$\int_{\mathbb{O}(p)} C_\lambda(XHYH^t)\, dH = \frac{C_\lambda(X)C_\lambda(Y)}{C_\lambda(I_p)} \tag{1.6}$$

where dH is normalized Haar measure.

By expanding the exponential and using (1.6) it follows that

$$\int_{\mathbb{O}(p)} \exp\left(z \operatorname{tr}(XHYH^t)\right) dH = \sum_{k \geq 0} \frac{z^k}{k!} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq p}} \frac{C_\lambda(X)C_\lambda(Y)}{C_\lambda(I_p)}. \tag{1.7}$$

We examine (1.7) for the special case ($|\rho| < 1$)

$$\Sigma = (1-\rho)I_p + \rho\, \mathbf{1} \otimes \mathbf{1} = \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & & & & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix}.$$

We have

$$\Sigma^{-1} = (1-\rho)^{-1} I_p - \frac{\rho}{(1-\rho)(1+(p-1)\rho)}\, \mathbf{1} \otimes \mathbf{1}$$

and

$$\det \Sigma = (1-\rho)^{p-1}\left(1+(p-1)\rho\right).$$

For this choice of Σ, let $Y = \alpha \mathbf{1} \otimes \mathbf{1}$ where $\alpha = \rho/((2(1-\rho)(1+(p-1)\rho))$, then

$$\begin{aligned}
\int_{\mathbb{O}(p)} e^{-\frac{1}{2}\operatorname{tr}(\Sigma^{-1}HLH^t)}\, dH &= e^{-\frac{1}{2(1-\rho)}\sum_j \lambda_j} \int_{\mathbb{O}(p)} e^{\operatorname{tr}(YHLH^t)}\, dH \\
&= e^{-\frac{1}{2(1-\rho)}\sum_j \lambda_j} \sum_{k \geq 0} \frac{C_{(k)}(\alpha\mathbf{1} \otimes \mathbf{1})C_{(k)}(\Lambda)}{k!C_{(k)}(I_p)}
\end{aligned}$$

[4]See, for example, Theorem 7.2.5 in [26].

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where we used the fact that the only partition $\lambda \vdash k$ for which $C_\lambda(Y)$ is nonzero is $\lambda = (k)$. And for this partition, $C_{(k)}(Y) = \alpha^k p^k$. Define the symmetric functions g_n[5] by

$$\prod_{j \geq 1}(1 - x_j y)^{-1/2} = \sum_{n \geq 0} g_n(x) y^n,$$

then it is known that [23]

$$C_{(k)}(L) = \frac{2^{2k}(k!)^2}{(2k)!}\, g_k(L).$$

Using the known value of $C_{(k)}(I_p)$ we find

$$\int_{\mathbb{O}(p)} e^{-\frac{1}{2}\operatorname{tr}(\Sigma^{-1}HLH^t)}\, dH = e^{-\frac{1}{2(1-\rho)}\sum_j \lambda_j} \sum_{k \geq 0} \frac{(\alpha p)^k}{(\frac{1}{2}p)_k}\, g_k(L)$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochammer symbol.

2 Edge Distribution Functions

2.1 Summary of Fredholm determinant representations

In this section we define three Fredholm determinants from which the edge eigenvalue distributions, for the three symmetry classes orthogonal, unitary and symplectic, will ensue. This section follows [31, 33, 36]; see also, [15, 16].

In the unitary case ($\beta = 2$), define the trace class operator K_2 on $L^2(s, \infty)$ with *Airy kernel*

$$K_{\mathrm{Ai}}(x, y) := \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y} = \int_0^\infty \mathrm{Ai}(x + z)\,\mathrm{Ai}(y + z)\, dz \quad (2.1)$$

and associated Fredholm determinant, $0 \leq \lambda \leq 1$,

$$D_2(s, \lambda) = \det(I - \lambda\, K_2). \tag{2.2}$$

Then we introduce the distribution functions

$$F_2(s) = F_2(s, 1) = D_2(s, 1), \tag{2.3}$$

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[5]In the theory of zonal polynomials, the g_n are the analogue of the complete symmetric functions h_n.

and for $m \geq 2$, the distribution functions $F_2(s, m)$ are defined recursively below by (3.9).

In the symplectic case $(\beta = 4)$, we define the trace class operator K_4 on $L^2(s, \infty) \oplus L^2(s, \infty)$ with matrix kernel

$$K_4(x, y) := \frac{1}{2} \left(\begin{array}{cc} S_4(x, y) & SD_4(x, y) \\ IS_4(x, y) & S_4(y, x) \end{array} \right) \tag{2.4}$$

where

$$\begin{aligned} S_4(x, y) &= K_{\mathrm{Ai}}(x, y) - \frac{1}{2} \mathrm{Ai}(x, y) \int_y^\infty \mathrm{Ai}(z)\, dz, \\ SD_4(x, y) &= -\partial_y K_{\mathrm{Ai}}(x, y) - \frac{1}{2} \mathrm{Ai}(x)\, \mathrm{Ai}(y), \\ IS_4(x, y) &= -\int_x^\infty K_{\mathrm{Ai}}(z, y)\, dz + \frac{1}{2} \int_x^\infty \mathrm{Ai}(z)\, dz \cdot \int_y^\infty \mathrm{Ai}(z), \end{aligned}$$

and the associated Fredholm determinant, $0 \leq \lambda \leq 1$,

$$D_4(s, \lambda) = \det(I - \lambda\, K_4). \tag{2.5}$$

Then we introduce the distribution functions (note the square root)

$$F_4(s) = F_4(s, 1) = \sqrt{D_4(s, 1)}, \tag{2.6}$$

and for $m \geq 2$, the distribution functions $F_4(s, m)$ are defined recursively below by (3.11).

In the orthogonal case $(\beta = 1)$, we introduce the matrix kernel

$$K_1(x, y) := \left(\begin{array}{cc} S_1(x, y) & SD_1(x, y) \\ IS_1(x, y) - \varepsilon(x, y) & S_1(y, x) \end{array} \right) \tag{2.7}$$

where

$$\begin{aligned} S_1(x, y) &= K_{\mathrm{Ai}}(x, y) - \frac{1}{2} \mathrm{Ai}(x) \left(1 - \int_y^\infty \mathrm{Ai}(z)\, dz \right), \\ SD_1(x, y) &= -\partial_y K_{\mathrm{Ai}}(x, y) - \frac{1}{2} \mathrm{Ai}(x)\, \mathrm{Ai}(y), \\ IS_1(x, y) &= -\int_x^\infty K_{\mathrm{Ai}}(z, y)\, dz + \frac{1}{2} \left(\int_y^x \mathrm{Ai}(z)\, dz + \int_x^\infty \mathrm{Ai}(z)\, dz \cdot \int_y^\infty \mathrm{Ai}(z)\, dz \right), \\ \varepsilon(x - y) &= \frac{1}{2} \mathrm{sgn}(x - y). \end{aligned}$$

The operator K_1 on $L^2(s, \infty) \oplus L^2(s, \infty)$ with this matrix kernel is *not* trace class due to the presence of ε. As discussed in [36], one must use the weighted space $L^2(\rho) \oplus L^2(\rho^{-1})$, $\rho^{-1} \in L^1$. Now the determinant is the 2-determinant,

$$D_1(s, \lambda) = \det{}_2(I - \lambda\, K_1 \chi_J) \tag{2.8}$$

where χ_J is the characteristic function of the interval (s, ∞). We introduce the distribution functions (again note the square root)

$$F_1(s) = F_1(s, 1) = \sqrt{D_1(s, 1)}\,, \tag{2.9}$$

and for $m \geq 2$, the distribution functions $F_1(s, m)$ are defined recursively below by (3.11). This is the first indication that the determinant $D_1(s, \lambda)$ might be more subtle than either $D_2(s, \lambda)$ or $D_4(s, \lambda)$.

2.2 Universality theorems

Suppose A is $W_p(n, I_p)$ with eigenvalues $\ell_1 > \cdots > \ell_p$. We define scaling constants

$$\mu_{np} = \left(\sqrt{n-1} + \sqrt{p}\right)^2, \quad \sigma_{np} = \left(\sqrt{n-1} + \sqrt{p}\right)\left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{p}}\right)^{1/3}.$$

The following theorem establishes, under the null hypothesis $\Sigma = I_p$, that the largest principal component variance, ℓ_1, converges in law to F_1.

Theorem 2.1 (Johnstone, [22]). *If $n, p \to \infty$ such that $n/p \to \gamma, 0 < \gamma < \infty$, then*

$$\frac{\ell_1 - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathscr{D}} F_1(s, 1).$$

Johnstone's theorem generalizes to the mth largest eigenvalue.

Theorem 2.2 (Soshnikov, [29]). *If $n, p \to \infty$ such that $n/p \to \gamma, 0 < \gamma < \infty$, then*

$$\frac{\ell_m - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathscr{D}} F_1(s, m),\ m = 1, 2, \ldots.$$

Soshnikov proved his result under the additional assumption $n - p = \mathrm{O}(p^{1/3})$. We remark that a straightforward generalization of Johnstone's proof [22] together with results of Dieng [10] show this restriction can be removed.

Subsequently, El Karoui [14] extended Theorem 2.2 to $0 < \gamma \leq \infty$. The extension to $\gamma = \infty$ is important for modern statistics where $p \gg n$ arises in applications.

Going further, Soshnikov lifted the Gaussian assumption, again establishing a F_1 universality theorem. In order to state the generalization precisely, let us redefine the $n \times p$ matrices $X = \{x_{i,j}\}$ such that $A = X^t X$ to satisfy

1. $\mathbb{E}(x_{ij}) = 0$, $\mathbb{E}(x_{ij}^2) = 1$.

2. The random variables x_{ij} have symmetric laws of distribution.

3. All even moments of x_{ij} are finite, and they decay at least as fast as a Gaussian at infinity: $\mathbb{E}(x_{ij}^{2m}) \leq (\text{const } m)^m$.

4. $n - p = \mathrm{O}(p^{1/3})$.

With these assumptions,

Theorem 2.3 (Soshnikov, [29]).

$$\frac{\ell_m - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathscr{D}} F_1(s, m), \; m = 1, 2, \ldots.$$

It is an important open problem to remove the restriction $n - p = \mathrm{O}(p^{1/3})$.

For real symmetric matrices, Deift and Gioev [8], building on the work of Widom [37], proved F_1 universality when the Gaussian weight function $\exp(-x^2)$ is replaced by $\exp(-V(x))$ where V is an even degree polynomial with positive leading coefficient.

Table 3 in Section 9 displays a comparison of the percentiles of the F_1 distribution with percentiles of empirical Wishart distributions. Here ℓ_j denotes the jth largest eigenvalue in the Wishart Ensemble. The percentiles in the ℓ_j columns were obtained by finding the ordinates corresponding to the F_1–percentiles listed in the first column, and computing the proportion of eigenvalues lying to the left of that ordinate in the empirical distributions for the ℓ_j. The bold entries correspond to the levels of confidence commonly used in statistical applications. The reader should compare Table 3 to similar ones in [14, 22].

3 Painlevé Representations: A Summary

The Gaussian β–ensembles are probability spaces on N-tuples of random variables $\{\ell_1, \ldots, \ell_N\}$, with joint density functions P_β given by[6]

$$P_\beta(\ell_1, \ldots, \ell_N) = P_\beta^{(N)}(\vec{\ell}\,) = C_\beta^{(N)} \exp\left[-\sum_{j=1}^{N} \ell_j^2\right] \prod_{j<k} |\ell_j - \ell_k|^\beta. \qquad (3.1)$$

The $C_\beta^{(N)}$ are normalization constants, given by

$$C_\beta^{(N)} = \pi^{-N/2} \, 2^{-N - \beta N(N-1)/4} \cdot \prod_{j=1}^{N} \frac{\Gamma(1+\gamma)\,\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}\,j)} \qquad (3.2)$$

By setting $\beta = 1, 2, 4$ we recover the (finite N) *Gaussian Orthogonal Ensemble* (GOE$_N$), *Gaussian Unitary Ensemble* (GUE$_N$), and *Gaussian Symplectic Ensemble* (GSE$_N$), respectively. For the remainder of the chapter we restrict to these three cases, and refer the reader to [12] for recent results on the general β case. Originally the ℓ_j are eigenvalues of randomly chosen matrices from corresponding matrix ensembles, so we will henceforth refer to them as eigenvalues. With the eigenvalues ordered so that $\ell_j \geq \ell_{j+1}$, define

$$\hat{\ell}_m^{(N)} = \frac{\ell_m - \sqrt{2N}}{2^{-1/2} \, N^{-1/6}}, \qquad (3.3)$$

to be the rescaled m^{th} eigenvalue measured from edge of spectrum. For the largest eigenvalue in the β–ensembles (proved only in the $\beta = 1, 2, 4$ cases) we have

$$\hat{\ell}_1^{(N)} \xrightarrow{\mathcal{D}} \hat{\ell}_1, \qquad (3.4)$$

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[6]In many places in the random matrix theory literature, the parameter β (times $1/2$) appears in front of the summation inside the exponential factor (3.1), in addition to being the power of the Vandermonde determinant. That convention originated in [24], and was justified by the alternative physical and very useful interpretation of (3.1) as a one–dimensional Coulomb gas model. In that language the potential $W = \frac{1}{2}\sum_i l_i^2 - \sum_{i<j} \ln|l_i - l_j|$ and $P_\beta^{(N)}(\vec{\ell}\,) = C\exp(-W/kT) = C\exp(-\beta W)$, so that $\beta = (kT)^{-1}$ plays the role of inverse temperature. However, by an appropriate choice of specialization in Selberg's integral, it is possible to remove the β in the exponential weight, at the cost of redefining the normalization constant $C_\beta^{(N)}$. We choose the latter convention in this work since we will not need the Coulomb gas analogy. Moreover, with computer simulations and statistical applications in mind, this will in our opinion make later choices of standard deviations, renormalizations, and scalings more transparent. It also allows us to dispose of the $\sqrt{2}$ that is often present in F_4.

whose law is given by the Tracy–Widom distributions.[6]

Theorem 3.1 (Tracy, Widom [31, 33]).

$$F_2(s) \;=\; \mathbb{P}_2(\hat{\ell}_1 \le s) = \exp\left[-\int_s^\infty (x-s)\,q^2(x)d\,x\right], \qquad (3.5)$$

$$F_1(s) \;=\; \mathbb{P}_1(\hat{\ell}_1 \le s) = (F_2(s))^{1/2}\,\exp\left[-\frac{1}{2}\int_s^\infty q(x)d\,x\right], \qquad (3.6)$$

$$F_4(s) \;=\; \mathbb{P}_4(\hat{\ell}_1 \le s) = (F_2(s))^{1/2}\,\cosh\left[-\frac{1}{2}\int_s^\infty q(x)d\,x\right]. \qquad (3.7)$$

The function q is the unique (see [6, 19]) solution to the Painlevé II equation

$$q'' = x\,q + 2\,q^3, \qquad (3.8)$$

such that $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$, where $\mathrm{Ai}(x)$ is the solution to the Airy equation which decays like $\frac{1}{2}\,\pi^{-1/2}\,x^{-1/4}\exp\left(-\frac{2}{3}\,x^{3/2}\right)$ at $+\infty$. The density functions f_β corresponding to the F_β are graphed in Figure 3.[7] Let $F_2(s,m)$
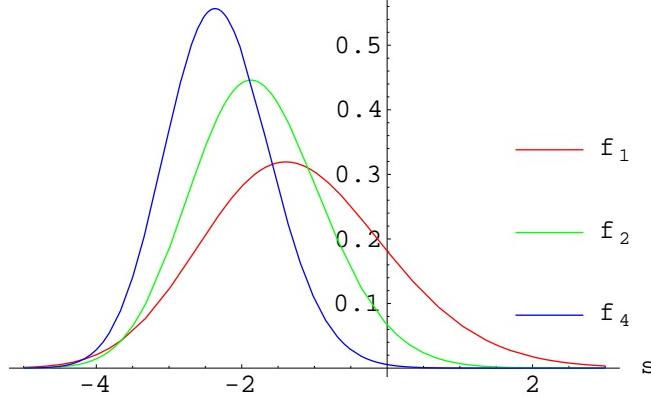
Figure 1: Tracy–Widom Density Functions

denote the distribution for the mth largest eigenvalue in GUE. Tracy and Widom showed [31] that if we define $F_2(s,0) \equiv 0$, then

$$F_2(s, m+1) - F_2(s, m) = \frac{(-1)^m}{m\,!}\,\frac{d^m}{d\,\lambda^m}\,D_2(s,\lambda)\Big|_{\lambda=1}, \quad m \ge 0, \qquad (3.9)$$

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[7]Actually, for $\beta = 4$, the density of $F_4(\sqrt{2}\,s)$ is graphed to agree with Mehta's original normalization [24] as well as with [33].

where (2.2) has the Painlevé representation

$$D_2(s, \lambda) = \exp\left[-\int_s^\infty (x - s)\, q^2(x, \lambda) d\,x\right], \qquad (3.10)$$

and $q(x, \lambda)$ is the solution to (3.8) such that $q(x, \lambda) \sim \sqrt{\lambda}\, \mathrm{Ai}(x)$ as $x \to \infty$. The same combinatorial argument used to obtain the recurrence (3.9) in the $\beta = 2$ case also works for the $\beta = 1, 4$ cases, leading to

$$F_\beta(s, m + 1) - F_\beta(s, m) = \frac{(-1)^m}{m\,!} \frac{d^m}{d\,\lambda^m}\, D_\beta^{1/2}(s, \lambda)\big|_{\lambda=1}, \quad m \geq 0, \beta = 1, 4, \qquad (3.11)$$

where $F_\beta(s, 0) \equiv 0$. Given the similarity in the arguments up to this point and comparing (3.10) to (3.5), it is natural to conjecture that $D_\beta(s, \lambda), \beta = 1, 4$, can be obtained simply by replacing $q(x)$ by $q(x, \lambda)$ in (3.6) and (3.7).

That this is not the case for $\beta = 1$ was shown by Dieng [10, 11]. A hint that $\beta = 1$ is different comes from the following interlacing theorem.

Theorem 3.2 (Baik, Rains [3]). *In the appropriate scaling limit, the distribution of the largest eigenvalue in GSE corresponds to that of the second largest in GOE. More generally, the joint distribution of every second eigenvalue in the GOE coincides with the joint distribution of all the eigenvalues in the GSE, with an appropriate number of eigenvalues.*

This interlacing property between GOE and GSE had long been in the literature, and had in fact been noticed by Mehta and Dyson [25]. In this context, Forrester and Rains [17] classified all weight functions for which alternate eigenvalues taken from an orthogonal ensemble form a corresponding symplectic ensemble, and similarly those for which alternate eigenvalues taken from a union of two orthogonal ensembles form an unitary ensemble. The following theorem gives explicit formulas for $D_1(s, \lambda)$ and $D_4(s, \lambda)$; and hence, from (3.11), a recursive procedure to determine $F_1(\cdot, m)$ and $F_4(\cdot, m)$ for $m \geq 2$.

Theorem 3.3 (Dieng [10, 11]). *In the edge scaling limit, the distributions for the mth largest eigenvalues in the GOE and GSE satisfy the recurrence (3.11) with[6]*

$$D_1(s, \lambda) = D_2(s, \tilde{\lambda})\, \frac{\lambda - 1 - \cosh \mu(s, \tilde{\lambda}) + \sqrt{\tilde{\lambda}}\, \sinh \mu(s, \tilde{\lambda})}{\lambda - 2}, \qquad (3.12)$$

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$$D_4(s,\lambda) = D_2(s,\lambda)\cosh^2\left(\frac{\mu(s,\lambda)}{2}\right), \tag{3.13}$$

where

$$\mu(s,\lambda) := \int_s^\infty q(x,\lambda)d\,x, \qquad \tilde{\lambda} := 2\,\lambda - \lambda^2, \tag{3.14}$$

and $q(x,\lambda)$ is the solution to (3.8) such that $q(x,\lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(x)$ as $x \to \infty$.

Note the appearance of $\tilde{\lambda}$ in the arguments on the right hand side of (3.12). In Fig. 2 we compare the densities $f_1(s,m)$, $m = 1,\ldots,4$, with finite N GOE simulations. This last theorem also provides a new proof of the Baik-Rains interlacing theorem.

Corollary 3.4 (Dieng [10, 11]).

$$F_4(s,m) = F_1(s,2\,m), \quad m \geq 1. \tag{3.15}$$

The proofs of these theorems occupy the bulk of the remaining part of the chapter. In the last section, we present an efficient numerical scheme to compute $F_\beta(s,m)$ and the associated density functions $f_\beta(s,m)$. We implemented this scheme using MATLAB$^{\mathrm{TM}}$,[8] and compared the results to simulated Wishart distributions.

4 Preliminaries

4.1 Determinant matters

We gather in this short section more or less classical results for further reference.

Theorem 4.1.

$$\prod_{0 \leq j < k \leq N}(x_j - x_k)^4 = \det\left(x_k^j \quad j\,x_k^{j-1}\right)_{\substack{j=0,\ldots,2\,N-1 \\ k=1,\ldots,N}}$$

---

[8]MATLAB$^{\mathrm{TM}}$ is a registered trademark of The MathWorks, Inc., 3 Apple Hill Drive, Natick, MA 01760-2098; Phone: 508-647-7000; Fax: 508-647-7001. Copies of the code are available by contacting the first author.
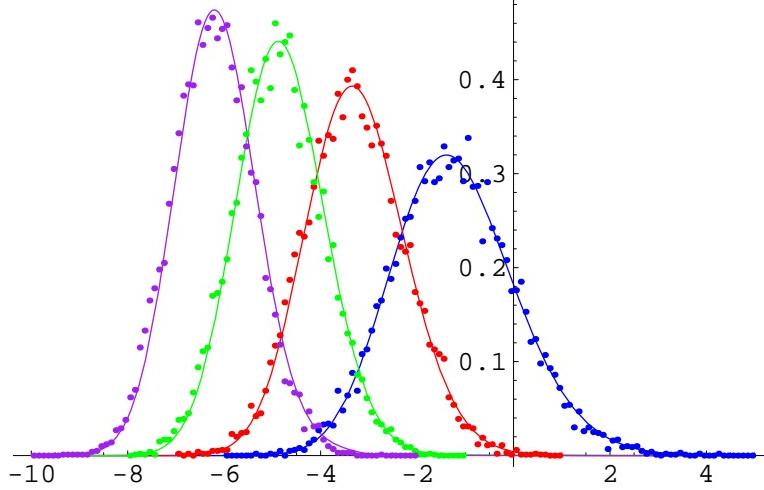
Figure 2: 10^4 realizations of $10^3 \times 10^3$ GOE matrices; the solid curves are, from right to left, the theoretical limiting densities for the first through fourth largest eigenvalue.

Theorem 4.2. *If A, B are Hilbert–Schmidt operators on a general[9] Hilbert space $\mathcal{H}$, then*

$$\det(I + A\,B) = \det(I + B\,A).$$

Theorem 4.3 (de Bruijn, [7]).

$$\int \cdots \int \det(\varphi_j(x_k))_{1 \le j,k \le N} \cdot \det(\psi_j(x_k))_{1 \le j,k \le N}\, d\,\mu(x_1) \cdots d\,\mu(x_N)$$
$$= N!\, \det\left(\int \varphi_j(x)\,\psi_k(x)\,d\,\mu(x)\right)_{1 \le j,k \le N}, \qquad (4.1)$$

$$\int_{x_1 \le \ldots \le x_N} \cdots \int \det(\varphi_j(x_k))_{1 \le j,k \le N} d\,\mu(x_1) \cdots d\,\mu(x_N)$$
$$= \mathrm{Pf}\left(\int \int \mathrm{sgn}(x - y)\varphi_j(x)\,\varphi_k(x)\,d\,\mu(x)\,d\,\mu(y)\right)_{1 \le j,k \le N}, \qquad (4.2)$$

---

[9]See [18] for proof.

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$$\int \cdots \int \det(\varphi_j(x_k) \quad \psi_j(x_k))_{\substack{1 \le j \le 2\,N \\ 1 \le k \le N}} d\,\mu(x_1) \cdots d\,\mu(x_N)$$

$$= (2\,N)!\,\mathrm{Pf}\left(\int \varphi_j(x)\psi_k(x) - \varphi_k(x)\psi_j(x)\,d\,\mu(x)\right)_{1 \le j,k \le 2\,N}, \tag{4.3}$$

where Pf denotes the Pfaffian. The last two integral identities were discovered by de Bruijn [7] in an attempt to generalize the first one. The first and last are valid in general measure spaces. In the second identity, the space needs to be ordered. In the last identity, the left hand side determinant is a $2\,N \times 2\,N$ determinant whose columns are alternating columns of the φ_j and ψ_j (i.e. the first four columns are $\{\varphi_j(x_1)\}$, $\{\psi_j(x_1)\}$, $\{\varphi_j(x_2)\}$, $\{\psi_j(x_2)\}$, respectively for $j = 1, \ldots, 2\,N$), hence the notation, and asymmetry in indexing.

A large portion of the foundational theory of random matrices, in the case of invariant measures, can be developed from Theorems 4.2 and 4.3 as was demonstrated in [34, 37].

4.2 Recursion formula for the eigenvalue distributions

With the joint density function defined as in (3.1), let J denote the interval (t, ∞), and $\chi = \chi_J(x)$ its characteristic function.[10] We denote by $\tilde{\chi} = 1 - \chi$ the characteristic function of the complement of J, and define $\tilde{\chi}_\lambda = 1 - \lambda\,\chi$. Furthermore, let $E_{\beta,N}(t, m)$ equal the probability that exactly the m largest eigenvalues of a matrix chosen at random from a (finite N) β–ensemble lie in J. We also define

$$G_{\beta,N}(t, \lambda) = \int \cdots \int_{x_i \in \mathbb{R}} \tilde{\chi}_\lambda(x_1) \cdots \tilde{\chi}_\lambda(x_N)\,P_\beta(x_1, \ldots, x_N)\,d\,x_1 \cdots d\,x_N. \tag{4.4}$$

For $\lambda = 1$ this is just $E_{\beta,N}(t, 0)$, the probability that no eigenvalues lie in (t, ∞), or equivalently the probability that the largest eigenvalue is less than t. In fact we will see in the following propositions that $G_{\beta,N}(t, \lambda)$ is in some sense a generating function for $E_{\beta,N}(t, m)$.

Proposition 4.4.

$$G_{\beta,N}(t, \lambda) = \sum_{k=0}^{N}(-\lambda)^k\binom{N}{k}\int \cdots \int_{x_i \in J} P_\beta(x_1, \ldots, x_N)\,d\,x_1 \cdots d\,x_N. \tag{4.5}$$

---

[10]Much of what is said here is still valid if J is taken to be a finite union of open intervals in $\mathbb{R}$ (see [32]). However, since we will only be interested in edge eigenvalues we restrict ourselves to (t, ∞) from here on.

Proof. Using the definition of the $\tilde{\chi}_\lambda(x_1)$ and multiplying out the integrand of (4.4) gives

$$G_{\beta,N}(t,\lambda) = \sum_{k=0}^{N}(-\lambda)^k \int \cdots \int_{x_i \in \mathbb{R}} e_k(\chi(x_1),\ldots,\chi(x_N))P_\beta(x_1,\ldots,x_N)\,dx_1\cdots dx_N,$$

where, in the notation of [30], $e_k = m_{1^k}$ is the kth elementary symmetric function. Indeed each term in the summation arises from picking k of the $\lambda\chi$-terms, each of which comes with a negative sign, and $N - k$ of the 1's. This explains the coefficient $(-\lambda)^k$. Moreover, it follows that e_k contains $\binom{N}{k}$ terms. Now the integrand is symmetric under permutations of the x_i. Also if $x_i \notin J$, all corresponding terms in the symmetric function are 0, and they are 1 otherwise. Therefore we can restrict the integration to $x_i \in J$, remove the characteristic functions (hence the symmetric function), and introduce the binomial coefficient to account for the identical terms up to permutation. $\square$

Proposition 4.5.

$$E_{\beta,N}(t,m) = \frac{(-1)^m}{m!}\frac{d^m}{d\lambda^m}\,G_{\beta,N}(t,\lambda)\Big|_{\lambda=1}, \qquad m \geq 0. \qquad (4.6)$$

Proof. This is proved by induction. As noted above, $E_{\beta,N}(t,0) = G_{\beta,N}(t,1)$ so it holds for the degenerate case $m = 0$. When $m = 1$ we have

$$-\frac{d}{d\lambda}\,G_{\beta,N}(t,\lambda)\Big|_{\lambda=1} = -\frac{d}{d\lambda}\int\cdots\int \tilde{\chi}_\lambda(x_1)\cdots\tilde{\chi}_\lambda(x_n)\,P_\beta^{(N)}(\vec{x})\,dx_1\cdots dx_n\Big|_{\lambda=1}$$

$$= -\sum_{j=1}^{N}-\int\cdots\int \tilde{\chi}(x_1)\cdots\tilde{\chi}(x_{j-1})\,\chi(x_j)\,\tilde{\chi}(x_{j+1})\cdots$$

$$\cdots\tilde{\chi}(x_N)\,P_\beta^{(N)}(\vec{x})\,dx_1\cdots dx_N.$$

The integrand is symmetric under permutations so we can make all terms look the same. There are $N = \binom{N}{1}$ of them so we get

$$-\frac{d}{d\lambda}\,G_{\beta,N}(t,\lambda)\Big|_{\lambda=1} = \binom{N}{1}\int\cdots\int \chi(x_1)\,\tilde{\chi}(x_2)\cdots\tilde{\chi}(x_N)\,P_\beta^{(N)}(\vec{x})\,dx_1\cdots dx_N\Big|_{\lambda=1}$$

$$= \binom{N}{1}\int\cdots\int \chi(x_1)\,\chi(x_2)\cdots\chi(x_N)\,P_\beta^{(N)}(\vec{x})\,dx_1\cdots dx_N$$

$$= E_{\beta,N}(t,1).$$

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When $m = 2$ then

$$\frac{1}{2}\left(-\frac{d}{d\lambda}\right)^2 G_{\beta,N}(t,\lambda)|_{\lambda=1}$$

$$= \frac{N}{2} \sum_{j=2}^{N} \int \cdots \int \chi(x_1)\tilde{\chi}(x_2)\cdots\tilde{\chi}(x_{j-1})\,\chi(x_j)\,\tilde{\chi}(x_{j+1})\cdots$$

$$\cdots \tilde{\chi}(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N\Big|_{\lambda=1}$$

$$= \frac{N(N-1)}{2} \int \cdots \int \chi(x_1)\,\chi(x_2)\,\tilde{\chi}(x_3)\cdots\tilde{\chi}(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N\Big|_{\lambda=1}$$

$$= \binom{N}{2} \int \cdots \int \chi(x_1)\,\chi(x_2)\,\tilde{\chi}(x_3)\cdots\tilde{\chi}(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N\Big|_{\lambda=1}$$

$$= \binom{N}{2} \int \cdots \int \chi(x_1)\,\chi(x_2)\,\chi(x_3)\cdots\chi(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N$$

$$= E_{\beta,N}(t,2),$$

where we used the previous case to get the first equality, and again the invariance of the integrand under symmetry to get the second equality. By induction then,

$$\frac{1}{m!}\left(-\frac{d}{d\lambda}\right)^m G_{\beta,N}(t,\lambda)|_{\lambda=1} =$$

$$= \frac{N(N-1)\cdots(N-m+2)}{m!} \sum_{j=m}^{N} \int \cdots \int \chi(x_1)\,\tilde{\chi}(x_2)\cdots$$

$$\cdots \tilde{\chi}(x_{j-1})\,\chi(x_j)\,\tilde{\chi}(x_{j+1})\cdots\tilde{\chi}(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N$$

$$= \frac{N(N-1)\cdots(N-m+1)}{m!} \int \cdots \int \chi(x_1)\cdots$$

$$\cdots \chi(x_m)\,\tilde{\chi}(x_{m+1})\cdots\tilde{\chi}(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N$$

$$= \binom{N}{m} \int \cdots \int \chi(x_1)\cdots\chi(x_m)\,\tilde{\chi}(x_{m+1})\cdots\tilde{\chi}(x_N)\,P_{\beta}^{(N)}(\vec{x})\,d\,x_1\cdots d\,x_N$$

$$= E_{\beta,N}(t,m).$$

$\square$

If we define $F_{\beta,N}(t,m)$ to be the distribution of the mth largest eigenvalue in the (finite N) β–ensemble, then the following probabilistic result is immediate from our definition of $E_{\beta,N}(t,m)$.

Corollary 4.6.

$$F_{\beta,N}(t, m+1) - F_{\beta,N}(t,m) = E_{\beta,N}(t,m) \qquad (4.7)$$

5 The Distribution of the mth Largest Eigenvalue in the GUE

5.1 The distribution function as a Fredholm determinant

We follow [34] for the derivations that follow. The GUE case corresponds to the specialization $\beta = 2$ in (3.1) so that

$$G_{2,N}(t,\lambda) = C_2^{(N)} \int \cdots \int_{x_i \in \mathbb{R}} \prod_{j<k} (x_j - x_k)^2 \prod_j^N w(x_j) \prod_j^N (1 + f(x_j)) \, d\,x_1 \cdots d\,x_N$$

$$(5.1)$$

where $w(x) = \exp\left(-x^2\right)$, $f(x) = -\lambda\,\chi_J(x)$, and $C_2^{(N)}$ depends only on N. In the steps that follow, additional constants depending solely on N (such as $N!$) which appear will be lumped into $C_2^{(N)}$. A probability argument will show that the resulting constant at the end of all calculations simply equals 1. Expressing the Vandermonde as a determinant

$$\prod_{1 \le j < k \le N} (x_j - x_k) = \det\left(x_k^j\right)_{\substack{j=0,\dots,N \\ k=1,\dots,N}} \qquad (5.2)$$

and using (4.1) with $\varphi_j(x) = \psi_j(x) = x^j$ and $d\,\mu(x) = w(x)(1 + f(x))$ yields

$$G_{2,N}(t,\lambda) = C_2^{(N)} \det\left(\int_{\mathbb{R}} x^{j+k}\, w(x)\,(1 + f(x))\, d\,x\right)_{j,k=0,\dots,N-1}. \qquad (5.3)$$

Let $\{\varphi_j(x)\}$ be the sequence obtained by orthonormalizing the sequence $\{x^j\, w^{1/2}(x)\}$. It follows that

$$\begin{aligned}
G_{2,N}(t,\lambda) &= C_2^{(N)} \det\left(\int_{\mathbb{R}} \varphi_j(x)\, \varphi_k(x)\, (1 + f(x))\, d\,x\right)_{j,k=0,\dots,N-1} & (5.4) \\
&= C_2^{(N)} \det\left(\delta_{j,k} + \int_{\mathbb{R}} \varphi_j(x)\, \varphi_k(x)\, f(x)\, d\,x\right)_{j,k=0,\dots,N-1} & (5.5)
\end{aligned}$$

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The last expression is of the form $\det(I+AB)$ for $A : L^2(\mathbb{R}) \to \mathbb{C}^N$ with kernel $A(j, x) = \varphi_j(x) f(x)$ whereas $B : \mathbb{C}^N \to L^2(\mathbb{R})$ with kernel $B(x, j) = \varphi_j(x)$. Note that $AB : \mathbb{C}^N \to \mathbb{C}^N$ has kernel

$$AB(j, k) = \int_{\mathbb{R}} \varphi_j(x) \, \varphi_k(x) \, f(x) \, d\,x \qquad (5.6)$$

whereas $BA : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has kernel

$$BA(x, y) = \sum_{k=0}^{N-1} \varphi_k(x) \, \varphi_k(y) := K_{2,N}(x, y). \qquad (5.7)$$

From Theorem 4.2 it follows that

$$G_{2,N}(t, \lambda) = C_2^{(N)} \det\left(I + K_{2,N} \, f\right), \qquad (5.8)$$

where $K_{2,N}$ has kernel $K_{2,N}(x, y)$ and $K_{2,N} \, f$ acts on a function by first multiplying it by f and acting on the product with $K_{2,N}$. From (5.1) we see that setting $f = 0$ in the last identity yields $C_2^{(N)} = 1$. Thus the above simplifies to

$$G_{2,N}(t, \lambda) = \det\left(I + K_{2,N} \, f\right). \qquad (5.9)$$

5.2 Edge scaling and differential equations

We specialize $w(x) = \exp\left(-x^2\right)$, $f(x) = -\lambda \chi_J(x)$, so that the $\{\varphi_j(x)\}$ are in fact the Hermite polynomials times the square root of the weight. Using the Plancherel-Rotach asymptotics of Hermite polynomials, it follows that in the *edge scaling limit*,

$$\lim_{N \to \infty} \frac{1}{2^{1/2} N^{1/6}} \, K_{N,2}\left(\sqrt{2N} + \frac{x}{2^{1/2} N^{1/6}}, \sqrt{2N} + \frac{y}{2^{1/2} N^{1/6}}\right) \chi_J\left(\sqrt{2N} + \frac{y}{2^{1/2} N^{1/6}}\right) \qquad (5.10)$$

is $K_{\mathrm{Ai}}(x, y)$ as defined in (2.1). As operators, the convergence is in trace class norm to K_2. (A proof of this last fact can be found in [36].) For notational convenience, we denote the corresponding operator K_2 by K in the rest of this subsection. It is convenient to view K as the integral operator on $\mathbb{R}$ with kernel

$$K(x, y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y} \chi_J(y), \qquad (5.11)$$

where $\varphi(x) = \sqrt{\lambda}\,\mathrm{Ai}(x)$, $\psi(x) = \sqrt{\lambda}\,\mathrm{Ai}'(x)$ and J is (s, ∞) with

$$t = \sqrt{2N} + \frac{s}{\sqrt{2}N^{1/6}}. \tag{5.12}$$

Note that although $K(x,y)$, φ and ψ are functions of λ as well, this dependence will not affect our calculations in what follows. Thus we omit it to avoid cumbersome notation. The Airy equation implies that φ and ψ satisfy the relations

$$
\begin{aligned}
\frac{d}{dx}\varphi &= \psi, \\
\frac{d}{dx}\psi &= x\,\varphi.
\end{aligned}
\tag{5.13}
$$

We define $D_{2,N}(s,\lambda)$ to be the Fredholm determinant $\det(I - K_{N,2})$. Thus in the edge scaling limit

$$\lim_{N\to\infty} D_{2,N}(s,\lambda) = D_2(s,\lambda).$$

We define the operator

$$R = (I - K)^{-1}K, \tag{5.14}$$

whose kernel we denote $R(x,y)$. Incidentally, we shall use the notation $\doteq$ in reference to an operator to mean "has kernel". For example $R \doteq R(x,y)$. We also let M stand for the operator whose action is multiplication by x. It is well known that

$$\frac{d}{ds}\log\det(I - K) = -R(s,s). \tag{5.15}$$

For functions f and g, we write $f \otimes g$ to denote the operator specified by

$$f \otimes g \doteq f(x)g(y), \tag{5.16}$$

and define

$$
\begin{aligned}
Q(x,s) &= Q(x) = \left((I-K)^{-1}\varphi\right)(x), & (5.17) \\
P(x,s) &= P(x) = \left((I-K)^{-1}\psi\right)(x). & (5.18)
\end{aligned}
$$

Then straightforward computation yields the following facts

$$
\begin{aligned}
[\,M\,,K\,] &= \varphi \otimes \psi - \psi \otimes \varphi, \\
[\,M\,,(I-K)^{-1}\,] &= (I-K)^{-1}\,[\,M\,,K\,]\,(I-K)^{-1} \\
&= Q \otimes P - P \otimes Q.
\end{aligned}
\tag{5.19}
$$

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On the other hand if $(I - K)^{-1} \doteq \rho(x, y)$, then

$$\rho(x, y) = \delta(x - y) + R(x, y), \tag{5.20}$$

and it follows that

$$\left[M, (I - K)^{-1} \right] \doteq (x - y)\rho(x, y) = (x - y)R(x, y). \tag{5.21}$$

Equating the two representation for the kernel of $\left[M, (I - K)^{-1} \right]$ yields

$$R(x, y) = \frac{Q(x)P(y) - P(x)Q(y)}{x - y}. \tag{5.22}$$

Taking the limit $y \to x$ and defining $q(s) = Q(s, s)$, $p(s) = P(s, s)$, we obtain

$$R(s, s) = Q'(s, s)\, p(s) - P'(s, s)\, q(s). \tag{5.23}$$

Let us now derive expressions for $Q'(x)$ and $P'(x)$. If we let the operator D stand for differentiation with respect to x,

$$\begin{aligned}
Q'(x, s) &= D\,(I - K)^{-1}\,\varphi \\
&= (I - K)^{-1}\,D\varphi + \left[D, (I - K)^{-1} \right]\varphi \\
&= (I - K)^{-1}\,\psi + \left[D, (I - K)^{-1} \right]\varphi \\
&= P(x) + \left[D, (I - K)^{-1} \right]\varphi.
\end{aligned} \tag{5.24}$$

We need the commutator

$$\left[D, (I - K)^{-1} \right] = (I - K)^{-1}\,[D, K]\,(I - K)^{-1}. \tag{5.25}$$

Integration by parts shows

$$[D, K] \doteq \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} \right) + K(x, s)\delta(y - s). \tag{5.26}$$

The δ function comes from differentiating the characteristic function χ. Moreover,

$$\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} \right) = \varphi(x)\varphi(y). \tag{5.27}$$

Thus

$$\left[D, (I - K)^{-1} \right] \doteq -Q(x)Q(y) + R(x, s)\rho(s, y). \tag{5.28}$$

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(Recall $(I - K)^{-1} \doteq \rho(x, y)$.) We now use this in (5.24) to obtain

$$
\begin{aligned}
Q'(x, s) &= P(x) - Q(x)\,(Q, \varphi) + R(x, s)q(s) \\
&= P(x) - Q(x)u(s) + R(x, s)q(s),
\end{aligned}
$$

where the inner product (Q, φ) is denoted by $u(s)$. Evaluating at $x = s$ gives

$$
Q'(s, s) = p(s) - q(s)u(s) + R(s, s)q(s). \tag{5.29}
$$

We now apply the same procedure to compute P'.

$$
\begin{aligned}
P'(x, s) &= D\,(I - K)^{-1}\,\psi \\
&= (I - K)^{-1}\,D\psi + \left[D, (I - K)^{-1}\right]\psi \\
&= M\,(I - K)^{-1}\,\varphi + \left[(I - K)^{-1}, M\right]\varphi + \left[D, (I - K)^{-1}\right]\psi \\
&= xQ(x) + (P \otimes Q - Q \otimes P)\,\varphi + (-Q \otimes Q)\psi + R(x, s)p(s) \\
&= xQ(x) + P(x)\,(Q, \varphi) - Q(x)\,(P, \varphi) - Q(x)\,(Q, \psi) + R(x, s)p(s) \\
&= xQ(x) - 2Q(x)v(s) + P(x)u(s) + R(x, s)p(s).
\end{aligned}
$$

Here $v = (P, \varphi) = (\psi, Q)$. Setting $x = s$ we obtain

$$
P'(s, s) = sq(s) + 2q(s)v(s) + p(s)u(s) + R(s, s)p(s). \tag{5.30}
$$

Using this and the expression for $Q'(s, s)$ in (5.23) gives

$$
R(s, s) = p^2 - sq^2 + 2q^2 v - 2pqu. \tag{5.31}
$$

Using the chain rule, we have

$$
\frac{d\,q}{d\,s} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial s}\right) Q(x, s)\,|_{x=s}. \tag{5.32}
$$

The first term is known. The partial with respect to s is

$$
\begin{aligned}
\frac{\partial Q(x, s)}{\partial s} &= (I - K)^{-1}\frac{\partial K}{\partial s}\,(I - K)^{-1}\,\varphi \\
&= -R(x, s)q(s),
\end{aligned}
$$

where we used the fact that

$$
\frac{\partial K}{\partial s} \doteq -K(x, s)\delta(y - s). \tag{5.33}
$$

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Adding the two partial derivatives and evaluating at $x = s$ gives

$$\frac{d\,q}{d\,s} = p - qu. \tag{5.34}$$

A similar calculation gives

$$\frac{d\,p}{d\,s} = sq - 2qv + pu. \tag{5.35}$$

We derive first order differential equations for u and v by differentiating the inner products. Recall that

$$u(s) = \int_s^\infty \varphi(x)Q(x, s)\,d\,x.$$

Thus

$$
\begin{aligned}
\frac{d\,u}{ds} &= -\varphi(s)q(s) + \int_s^\infty \varphi(x)\frac{\partial Q(x, s)}{\partial s}\,d\,x \\
&= -\left(\varphi(s) + \int_s^\infty R(s, x)\varphi(x)\,d\,x\right)q(s) \\
&= -(I - K)^{-1}\,\varphi(s)\,q(s) \\
&= -q^2.
\end{aligned}
$$

Similarly,

$$\frac{d\,v}{d\,s} = -pq. \tag{5.36}$$

From the first order differential equations for q, u and v it follows immediately that the derivative of $u^2 - 2v - q^2$ is zero. Examining the behavior near $s = \infty$ to check that the constant of integration is zero then gives

$$u^2 - 2v = q^2. \tag{5.37}$$

We now differentiate (5.34) with respect to s, use the first order differential equations for p and u, and then the first integral to deduce that q satisfies the Painlevé II equation (3.8). Checking the asymptotics of the Fredholm determinant $\det(I - K)$ for large s shows we want the solution with boundary condition

$$q(s, \lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(s) \qquad \text{as} \qquad s \to \infty. \tag{5.38}$$

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That a solution q exists and is unique follows from the representation of the Fredholm determinant in terms of it. Independent proofs of this, as well as the asymptotics as $s \to \infty$ were given by [19], [6], [9]. Since $[\, D \,, (I - K)^{-1}\,] \doteq (\partial/\partial x \,+\, \partial/\partial y)\, R(x, y)$, (5.28) says

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R(x, y) = -Q(x)Q(y) + R(x, s)\rho(s, y). \qquad (5.39)$$

In computing $\partial Q(x, s)/\partial s$ we showed that

$$\frac{\partial}{\partial s} (I - K)^{-1} \doteq \frac{\partial}{\partial s} R(x, y) = -R(x, s)\rho(s, y). \qquad (5.40)$$

Adding these two expressions,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial s} \right) R(x, y) = -Q(x)\, Q(y), \qquad (5.41)$$

and then evaluating at $x = y = s$ gives

$$\frac{d}{d\, s} R(s, s) = -q^2. \qquad (5.42)$$

Integration (and recalling (5.15)) gives,

$$\frac{d}{d\, s} \log \det (I - K) = - \int_s^\infty q^2(x, \lambda)\, d\, x; \qquad (5.43)$$

and hence,

$$\log \det (I - K) = - \int_s^\infty \left(\int_y^\infty q^2(x, \lambda)\, d\, x \right) d\, y = - \int_s^\infty (x - s)q^2(x, \lambda)\, d\, x. \qquad (5.44)$$

To summarize, we have shown that $D_2(s, \lambda)$ has the Painlevé representation (3.10) where q satisfies the Painlevé II equation (3.8) subject to the boundary condition (5.38).

6 The Distribution of the mth Largest Eigenvalue in the GSE

6.1 The distribution function as a Fredholm determinant

The GSE corresponds case corresponds to the specialization $\beta = 4$ in (3.1) so that

$$G_{4,N}(t, \lambda) = C_4^{(N)} \int \cdots \int_{x_i \in \mathbb{R}} \prod_{j<k} (x_j - x_k)^4 \prod_j^N w(x_j) \prod_j^N (1 + f(x_j)) \, dx_1 \cdots dx_N$$

(6.1)

where $w(x) = \exp(-x^2)$, $f(x) = -\lambda \chi_J(x)$, and $C_4^{(N)}$ depends only on N. As in the GUE case, we will absorb into $C_4^{(N)}$ any constants depending only on N that appear in the derivation. A simple argument at the end will show that the final constant is 1. These calculations follow [34]. By (4.1), $G_{4,N}(t, \lambda)$ is given by the integral

$$C_4^{(N)} \int \cdots \int_{x_i \in \mathbb{R}} \det \left(x_k^j \quad j \, x_k^{j-1} \right)_{\substack{j=0,\ldots,2N-1 \\ k=1,\ldots,N}} \prod_{i=1}^N w(x_i) \prod_{i=1}^N (1 + f(x_i)) \, dx_1 \cdots dx_N$$

which, if we define $\varphi_j(x) = x^{j-1} \, w(x) \, (1 + f(x))$ and $\psi_j(x) = (j-1) \, x^{j-2}$ and use the linearity of the determinant, becomes

$$G_{4,N}(t, \lambda) = C_4^{(N)} \int \cdots \int_{x_i \in \mathbb{R}} \det \left(\varphi_j(x_k) \quad \psi(x_k) \right)_{\substack{1 \leq j \leq 2N \\ 1 \leq k \leq N}} dx_1 \cdots dx_N.$$

Now using (4.3), we obtain

$$\begin{aligned}
G_{4,N}(t, \lambda) &= C_4^{(N)} \, \text{Pf} \left(\int \varphi_j(x)\psi_k x - \varphi_k(x)\psi_j(x) \, dx \right)_{1 \leq j,k \leq 2N} \\
&= C_4^{(N)} \, \text{Pf} \left(\int (k-j) \, x^{j+k-3} \, w(x) \, (1 + f(x)) \, dx \right)_{1 \leq j,k \leq 2N} \\
&= C_4^{(N)} \, \text{Pf} \left(\int (k-j) \, x^{j+k-1} \, w(x) \, (1 + f(x)) \, dx \right)_{0 \leq j,k \leq 2N-1},
\end{aligned}$$

where we let $k \to k+1$ and $j \to j+1$ in the last line. Remembering that the square of a Pfaffian is a determinant, we obtain

$$G_{4,N}^2(t, \lambda) = C_4^{(N)} \det \left(\int (k-j)\, x^{j+k-1}\, w(x)\, (1 + f(x))\, d\,x \right)_{0 \le j, k \le 2\,N-1}.$$

Row operations on the matrix do not change the determinant, so we can replace $\{x^j\}$ by an arbitrary sequence $\{p_j(x)\}$ of polynomials of degree j obtained by adding rows to each other. Note that the general (j, k) element in the matrix can be written as

$$\left[\left(\frac{d}{d\,x} x^k \right) x^j - \left(\frac{d}{d\,x} x^j \right) x^k \right]\, w(x)\, (1 + f(x)).$$

Thus when we add rows to each other the polynomials we obtain will have the same general form (the derivatives factor). Therefore we can assume without loss of generality that $G_{4,N}^2(t, \lambda)$ equals

$$C_4^{(N)} \det \left(\int \left[p_j(x)\, p_k'(x) - p_j'(x)\, p_k(x) \right]\, w(x)\, (1 + f(x))\, d\,x \right)_{0 \le j, k \le 2\,N-1},$$

where the sequence $\{p_j(x)\}$ of polynomials of degree j is arbitrary. Let $\psi_j = p_j\, w^{1/2}$ so that $p_j = \psi_j\, w^{-1/2}$. Substituting this into the above formula and simplifying, we obtain

$$
\begin{aligned}
G_{4,N}^2(t, \lambda) &= C_4^{(N)} \det \left[\int \left(\left(\psi_j(x)\, \psi_k'(x) - \psi_k(x)\, \psi_j'(x) \right) (1 + f(x)) \right)\, d\,x \right]_{0 \le j, k \le 2\,N-1} \\
&= C_4^{(N)} \det \left[M + L \right] = C_4^{(N)} \det[M]\, \det[I + M^{-1} \cdot L],
\end{aligned}
$$

where M, L are matrices given by

$$M = \left(\int \left(\psi_j(x)\, \psi_k'(x) - \psi_k(x)\, \psi_j'(x) \right)\, d\,x \right)_{0 \le j, k \le 2\,N-1},$$

$$L = \left(\int \left(\psi_j(x)\, \psi_k'(x) - \psi_k(x)\, \psi_j'(x) \right)\, f(x)\, d\,x \right)_{0 \le j, k \le 2\,N-1}.$$

Note that $\det[M]$ is a constant which depends only on N so we can absorb it into $C_4^{(N)}$. Also if we denote

$$M^{-1} = \{\mu_{jk}\}_{0 \le j, k \le 2\,N-1}, \qquad \eta_j = \sum_{k=0}^{2N-1} \mu_{jk} \psi_k(x),$$

it follows that

$$M^{-1} \cdot N = \left\{ \int \left(\eta_j(x) \, \psi_k'(x) - \eta_j'(x) \, \psi_k(x) \right) f(x) \, d\,x \right\}_{0 \le j, k \le 2\,N-1}.$$

Let $A : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{C}^{2N}$ be the operator defined by the $2N \times 2$ matrix

$$A(x) = \begin{pmatrix} \eta_0(x) & -\eta_0'(x) \\ \eta_1(x) & -\eta_1'(x) \\ \vdots & \vdots \end{pmatrix}.$$

Thus if

$$g = \begin{pmatrix} g_0(x) \\ g_1(x) \end{pmatrix} \in L^2(\mathbb{R}) \times L^2(\mathbb{R}),$$

we have

$$A\,g = A(x)\,g = \begin{pmatrix} \int (\eta_0 g_0 - \eta_0' g_1) \, d\,x \\ \int (\eta_1 g_0 - \eta_1' g_1) \, d\,x \\ \vdots \end{pmatrix} \in \mathbb{C}^{2N}.$$

Similarly we define $B : \mathbb{C}^{2n} \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ given by the $2 \times 2n$ matrix

$$B(x) = f \cdot \begin{pmatrix} \psi_0'(x) & \psi_1'(x) & \cdots \\ \psi_0(x) & \psi_1(x) & \cdots \end{pmatrix}.$$

Explicitly if

$$\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \end{pmatrix} \in \mathbb{C}^{2N},$$

then

$$B\alpha = B(x) \cdot \alpha = \begin{pmatrix} f \sum_{i=0}^{2N-1} \alpha_i \psi_i' \\ f \sum_{i=0}^{2N-1} \alpha_i \psi_i \end{pmatrix} \in L^2 \times L^2.$$

Observe that $M^{-1} \cdot L = AB : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Indeed

$$AB\alpha = \begin{pmatrix} \sum_{i=0}^{2N-1} \left[\int (\eta_0 \psi_i' - \eta_0' \psi_i) f \, dx \right] \alpha_i \\ \sum_{i=0}^{2N-1} \left[\int (\eta_1 \psi_i' - \eta_1' \psi_i) f \, dx \right] \alpha_i \\ \vdots \end{pmatrix}$$

$$= \left\{ \int (\eta_j \psi_k' - \eta_j' \psi_k) f \, dx \right\} \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \end{pmatrix} = (M^{-1} \cdot L) \alpha.$$

Therefore, by (4.2)

$$G_{4,N}^2(t, \lambda) = C_4^{(N)} \det(I + M^{-1} \cdot L) = C_4^{(N)} \det(I + AB) = C_4^{(N)} \det(I + BA)$$

where $BA : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. From our definition of A and B it follows that

$$BAg = \begin{pmatrix} f \sum_{i=0}^{2n-1} \psi_i' \left(\int (\eta_i g_0 - \eta_i' g_1) \, dx \right) \\ f \sum_{i=0}^{2N-1} \psi_i' \left(\int (\eta_i g_0 - \eta_i' g_1) \, dx \right) \end{pmatrix}$$

$$= f \begin{pmatrix} \int \sum_{i=0}^{2N-1} \psi_i'(x) \eta_i(y) g_0(y) \, dy - \int \sum_{i=0}^{2N-1} \psi_i'(x) \eta_i'(y) g_1(y) \, dy \\ \int \sum_{i=0}^{2N-1} \psi_i(x) \eta_i(y) g_0(y) \, dy - \int \sum_{i=0}^{2N-1} \psi_i(x) \eta_i'(y) g_1(y) \, dy \end{pmatrix}$$

$$= f K_{4,N} \, g,$$

where $K_{4,N}$ is the integral operator with matrix kernel

$$K_{4,N}(x,y) = \begin{pmatrix} \sum_{i=0}^{2N-1} \psi_i'(x) \eta_i(y) & -\sum_{i=0}^{2N-1} \psi_i'(x) \eta_i'(y) \\ \sum_{i=0}^{2N-1} \psi_i(x) \eta_i(y) & -\sum_{i=0}^{2N-1} \psi_i(x) \eta_i'(y) \end{pmatrix}.$$

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Recall that $\eta_j(x) = \sum_{k=0}^{2N-1} \mu_{jk}\psi_k(x)$ so that

$$K_{4,N}(x,y) = \begin{pmatrix} \displaystyle\sum_{j,k=0}^{2N-1} \psi_j'(x)\mu_{jk}\psi_k(y) & -\displaystyle\sum_{j,k=0}^{2N-1} \psi_j'(x)\mu_{jk}\psi_k'(y) \\ \displaystyle\sum_{j,k=0}^{2N-1} \psi_j(x)\mu_{jk}\psi_k(y) & -\displaystyle\sum_{j,k=0}^{2N-1} \psi_j(x)\mu_{jk}\psi_k'(y) \end{pmatrix}.$$

Define ϵ to be the following integral operator

$$\epsilon \doteq \epsilon(x-y) = \begin{cases} \frac{1}{2} & \text{if } x > y, \\ -\frac{1}{2} & \text{if } x < y. \end{cases} \tag{6.2}$$

As before, let D denote the operator that acts by differentiation with respect to x. The fundamental theorem of calculus implies that $D\,\epsilon = \epsilon\,D = I$. We also define

$$S_N(x,y) = \sum_{j,k=0}^{2N-1} \psi_j'(x)\mu_{jk}\psi_k(y).$$

Since M is antisymmetric,

$$S_N(y,x) = \sum_{j,k=0}^{2N-1} \psi_j'(y)\mu_{jk}\psi_k(x) = -\sum_{j,k=0}^{2N-1} \psi_j'(y)\mu_{kj}\psi_k(x) = -\sum_{j,k=0}^{2N-1} \psi_j(y)\mu_{kj}\psi_k'(x),$$

after re-indexing. Note that

$$\epsilon\, S_N(x,y) = \sum_{j,k=0}^{2N-1} \epsilon\, D\,\psi_j(x)\mu_{jk}\psi_k(y) = \sum_{j,k=0}^{2N-1} \psi_j(x)\mu_{jk}\psi_k(y),$$

and

$$-\frac{d}{d\,y} S_N(x,y) = \sum_{j,k=0}^{2N-1} \psi_j'(x)\mu_{jk}\psi_k'(y).$$

Thus we can now write succinctly

$$K_N(x,y) = \begin{pmatrix} S_N(x,y) & -\frac{d}{d\,y} S_N(x,y) \\ \epsilon\, S_N(x,y) & S_N(y,x) \end{pmatrix}. \tag{6.3}$$

To summarize, we have shown that $G_{4,N}^2(t, \lambda) = C_4^{(N)} \det(I - K_{4,N}f)$. Setting $f \equiv 0$ on both sides (where the original definition of $G_{4,N}(t, \lambda)$ as an integral is used on the left) shows that $C_4^{(N)} = 1$. Thus

$$G_{4,N}(t, \lambda) = \sqrt{D_{4,N}(t, \lambda)}, \tag{6.4}$$

where we define

$$D_{4,N}(t, \lambda) = \det(I + K_{4,N}f), \tag{6.5}$$

and $K_{4,N}$ is the integral operator with matrix kernel (6.3).

6.2 Gaussian specialization

We would like to specialize the above results to the case of a Gaussian weight function

$$w(x) = \exp\left(x^2\right) \tag{6.6}$$

and indicator function

$$f(x) = -\lambda \chi_J \quad , \quad J = (t, \infty).$$

We want the matrix

$$M = \left\{ \int \left(\psi_j(x) \, \psi_k'(x) - \psi_k(x) \, \psi_j'(x) \right) \, dx \right\}_{0 \leq j,k \leq 2N-1}$$

to be the direct sum of N copies of

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that the formulas are the simplest possible, since then μ_{jk} can only be 0 or ± 1. In that case M would be skew–symmetric so that $M^{-1} = -M$. In terms of the integrals defining the entries of M this means that we would like to have

$$\int \left(\psi_{2j}(x) \frac{d}{dx} \psi_{2k+1}(x) - \psi_{2k+1}(x) \frac{d}{dx} \psi_{2j}(x) \right) \, dx = \delta_{j,k},$$

$$\int \left(\psi_{2j+1}(x) \frac{d}{dx} \psi_{2k}(x) - \psi_{2k}(x) \frac{d}{dx} \psi_{2j+1}(x) \right) \, dx = -\delta_{j,k}$$

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and otherwise

$$\int \left(\psi_j(x) \frac{d}{dx} \psi_k(x) - \psi_j(x) \frac{d}{dx} \psi_k(x) \right) dx = 0.$$

It is easier to treat this last case if we replace it with three non-exclusive conditions

$$\int \left(\psi_{2j}(x) \frac{d}{dx} \psi_{2k}(x) - \psi_{2k}(x) \frac{d}{dx} \psi_{2j}(x) \right) dx = 0,$$

$$\int \left(\psi_{2j+1}(x) \frac{d}{dx} \psi_{2k+1}(x) - \psi_{2k+1}(x) \frac{d}{dx} \psi_{2j+1}(x) \right) dx = 0,$$

(so when the parity is the same for j, k, which takes care of diagonal entries, among others) and

$$\int \left(\psi_j(x) \frac{d}{dx} \psi_k(x) - \psi_j(x) \frac{d}{dx} \psi_k(x) \right) dx = 0,$$

whenever $|j - k| > 1$, which targets entries outside of the tridiagonal. Define

$$\varphi_k(x) = \frac{1}{c_k} H_k(x) \exp(-x^2/2) \quad \text{for} \quad c_k = \sqrt{2^k k! \sqrt{\pi}} \tag{6.7}$$

where the H_k are the usual Hermite polynomials defined by the orthogonality condition

$$\int_{\mathbb{R}} H_j(x) H_k(x) e^{-x^2} dx = c_j^2 \delta_{j,k}.$$

Then it follows that

$$\int_{\mathbb{R}} \varphi_j(x) \varphi_k(x) dx = \delta_{j,k}.$$

Now let

$$\psi_{2j+1}(x) = \frac{1}{\sqrt{2}} \varphi_{2j+1}(x) \qquad \psi_{2j}(x) = -\frac{1}{\sqrt{2}} \epsilon \varphi_{2j+1}(x)$$

This definition satisfies our earlier requirement that $\psi_j = p_j w^{1/2}$ with w defined in (6.6). In particular we have in this case

$$p_{2j+1}(x) = \frac{1}{c_j \sqrt{2}} H_{2j+1}(x).$$

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Let ϵ as in (6.2), and D denote the operator that acts by differentiation with respect to x as before, so that $D\,\epsilon = \epsilon\,D = I$. It follows that

$$\int_{\mathbb{R}} \left[\psi_{2j}(x) \frac{d}{d\,x} \psi_{2k+1}(x) - \psi_{2k+1}(x) \frac{d}{d\,x} \psi_{2j}(x) \right] d\,x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[-\epsilon\,\varphi_{2j+1}(x) \frac{d}{d\,x} \varphi_{2k+1}(x) + \varphi_{2k+1}(x) \frac{d}{d\,x} \epsilon\,\varphi_{2j+1}(x) \right] d\,x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[-\epsilon\,\varphi_{2j+1}(x) \frac{d}{d\,x} \varphi_{2k+1}(x) + \varphi_{2k+1}(x)\,\varphi_{2j+1}(x) \right] d\,x$$

We integrate the first term by parts and use the fact that

$$\frac{d}{d\,x} \epsilon\,\varphi_j(x) = \varphi_j(x)$$

and also that φ_j vanishes at the boundary (i.e. $\varphi_j(\pm\infty) = 0$) to obtain

$$\int_{\mathbb{R}} \left[\psi_{2j}(x) \frac{d}{d\,x} \psi_{2k+1}(x) - \psi_{2k+1}(x) \frac{d}{d\,x} \psi_{2j}(x) \right] d\,x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[-\epsilon\,\varphi_{2j+1}(x) \frac{d}{d\,x} \varphi_{2k+1}(x) + \varphi_{2k+1}(x)\,\varphi_{2j+1}(x) \right] d\,x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[\varphi_{2j+1}(x)\,\varphi_{2k+1}(x) + \varphi_{2k+1}(x)\,\varphi_{2j+1}(x) \right] d\,x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[\varphi_{2j+1}(x)\,\varphi_{2k+1}(x) + \varphi_{2k+1}(x)\,\varphi_{2j+1}(x) \right] d\,x$$

$$= \frac{1}{2} \left(\delta_{j,k} + \delta_{j,k} \right)$$

$$= \delta_{j,k},$$

as desired. Similarly

$$\int_{\mathbb{R}} \left[\psi_{2j+1}(x) \frac{d}{d\,x} \psi_{2k}(x) - \psi_{2k}(x) \frac{d}{d\,x} \psi_{2j+1}(x) \right] d\,x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[-\varphi_{2j+1}(x) \frac{d}{d\,x} \epsilon\,\varphi_{2k+1}(x) + \epsilon\,\varphi_{2k+1}(x) \frac{d}{d\,x} \varphi_{2j+1}(x) \right] d\,x$$

$$= -\delta_{j,k}.$$

Moreover,

$$p_{2j+1}(x) = \frac{1}{c_j\sqrt{2}} H_{2j+1}(x)$$

is certainly an odd function, being the multiple of and odd Hermite polynomial. On the other hand, one easily checks that ϵ maps odd functions to even functions on $L^2(\mathbb{R})$. Therefore

$$p_{2j}(x) = -\frac{1}{c_j\sqrt{2}}\,\epsilon\,H_{2j+1}(x)$$

is an even function, and it follows that

$$\int_{\mathbb{R}}\left[\psi_{2k}(x)\frac{d}{dx}\,\psi_{2j}(x) - \psi_{2j}(x)\frac{d}{dx}\,\psi_{2k}(x)\right]dx$$
$$= \int_{\mathbb{R}}\left[p_{2j}(x)\frac{d}{dx}\,p_{2k}(x) - p_{2k}(x)\frac{d}{dx}\,p_{2j}(x)\right]w(x)\,dx$$
$$= 0,$$

since both terms in the integrand are odd functions, and the weight function is even. Similarly,

$$\int_{\mathbb{R}}\left[\psi_{2k+1}(x)\frac{d}{dx}\,\psi_{2j+1}(x) - \psi_{2j+1}(x)\frac{d}{dx}\,\psi_{2k+1}(x)\right]dx$$
$$= \int_{\mathbb{R}}\left[p_{2j+1}(x)\frac{d}{dx}\,p_{2k+1}(x) - p_{2k+1}(x)\frac{d}{dx}\,p_{2j+1}(x)\right]w(x)\,dx$$
$$= 0.$$

Finally it is easy to see that if $|j - k| > 1$ then

$$\int_{\mathbb{R}}\left[\psi_j(x)\frac{d}{dx}\,\psi_k(x) - \psi_j(x)\frac{d}{dx}\,\psi_k(x)\right]dx = 0.$$

Indeed both differentiation and the action of ϵ can only "shift" the indices by 1. Thus by orthogonality of the φ_j, this integral will always be 0. Hence by choosing

$$\psi_{2j+1}(x) = \frac{1}{\sqrt{2}}\,\varphi_{2j+1}(x), \qquad \psi_{2j}(x) = -\frac{1}{\sqrt{2}}\,\epsilon\,\varphi_{2j+1}(x),$$

we force the matrix

$$M = \left\{\int_{\mathbb{R}}\left(\psi_j(x)\,\psi_k'(x) - \psi_k(x)\,\psi_j'(x)\right)dx\right\}_{0\le j,k\le 2\,n-1}$$

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to be the direct sum of N copies of

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence $M^{-1} = -M$ where $M^{-1} = \{\mu_{j,k}\}_{j,k=0,2N-1}$. Moreover, with our above choice, $\mu_{j,k} = 0$ if j, k have the same parity or $|j - k| > 1$, and $\mu_{2k,2j+1} = \delta_{jk} = -\mu_{2j+1,2k}$ for $j, k = 0. \ldots, N-1$. Therefore

$$S_N(x, y) = -\sum_{j,k=0}^{2N-1} \psi'_j(x)\mu_{jk}\psi_k(y)$$

$$= -\sum_{j=0}^{N-1} \psi'_{2j}(x)\,\psi_{2j+1}(y) + \sum_{j=0}^{N-1} \psi'_{2j+1}(x)\,\psi_{2j}(y)$$

$$= \frac{1}{2}\left[\sum_{j=0}^{N-1} \varphi_{2j+1}(x)\,\varphi_{2j+1}(y) - \sum_{j=0}^{N-1} \varphi'_{2j+1}(x)\,\epsilon\,\varphi_{2j+1}(y)\right].$$

Recall that the H_j satisfy the differentiation formulas (see for example [2], p. 280)

$$H'_j(x) = 2\,x\,H_j(x) - H_{j-1}(x) \quad j = 1, 2, \ldots \tag{6.8}$$

$$H'_j(x) = 2\,j\,H_{j-1}(x) \quad j = 1, 2, \ldots \tag{6.9}$$

Combining (6.7) and (6.8) yields

$$\varphi'_j(x) = x\,\varphi_j(x) - \frac{c_{j+1}}{c_j}\,\varphi_{j+1}(x). \tag{6.10}$$

Similarly, from (6.7) and (6.9) we have

$$\varphi'_j(x) = -x\,\varphi_j(x) + 2j\,\frac{c_{j-1}}{c_j}\,\varphi_{j-1}(x). \tag{6.11}$$

Combining (6.10) and (6.11), we obtain

$$\varphi'_j(x) = \sqrt{\frac{j}{2}}\,\varphi_{j-1}(x) - \sqrt{\frac{j+1}{2}}\,\varphi_{j+1}(x). \tag{6.12}$$

Let $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \end{pmatrix}$ and $\varphi' = \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \\ \vdots \end{pmatrix}$. Then we can rewrite (6.12) as

$$\varphi' = A\,\varphi$$

where $A = \{a_{j,k}\}$ is the infinite antisymmetric tridiagonal matrix with $a_{j,j-1} = \sqrt{\frac{j}{2}}$. Hence,

$$\varphi'_j(x) = \sum_{k \geq 0} a_{jk}\varphi_k(x).$$

Moreover, using the fact that $D\,\epsilon = \epsilon\,D = I$ we also have

$$\varphi_j(x) = \epsilon\,\varphi'_j(x) = \epsilon\left(\sum_{k \geq 0} a_{jk}\varphi_k(x)\right)$$

$$= \sum_{k \geq 0} a_{jk}\epsilon\,\varphi_k(x).$$

Combining the above results, we have

$$\sum_{j=0}^{N-1} \varphi'_{2j+1}(x)\,\epsilon\,\varphi_{2j+1}(y) = \sum_{j=0}^{N-1}\sum_{k \geq 0} a_{2j+1,k}\varphi_k(x)\,\epsilon\,\varphi_{2j+1}(x)$$

$$= -\sum_{j=0}^{N-1}\sum_{k \geq 0} a_{k,2j+1}\varphi_k(x)\,\epsilon\,\varphi_{2j+1}(x).$$

Note that $a_{k,2j+1} = 0$ unless $|k - (2j+1)| = 1$, that is unless k is even. Thus we can rewrite the sum as

$$\sum_{j=0}^{N-1} \varphi'_{2j+1}(x)\,\epsilon\,\varphi_{2j+1}(y) = -\sum_{\substack{k,j \geq 0 \\ k\ \text{even} \\ k \leq 2N}} a_{k,j}\varphi_k(x)\,\epsilon\,\varphi_j(y) - a_{2N,2N+1}\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)$$

$$= -\sum_{\substack{k \geq 0 \\ k\ \text{even} \\ k \leq 2N}} \varphi_k(x)\sum_{j \geq 0} a_{k,j}\epsilon\,\varphi_j(y) + a_{2N,2N+1}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)$$

where the last term takes care of the fact that we are counting an extra term in the sum that was not present before. The sum over j on the right is just

$\varphi_k(y)$, and $a_{2N,2N+1} = -\sqrt{\frac{2N+1}{2}}$. Therefore

$$\sum_{j=0}^{N-1} \varphi'_{2j+1}(x)\,\epsilon\,\varphi_{2j+1}(y) = \sum_{\substack{k\geq 0 \\ k \text{ even} \\ k\leq 2N}} \varphi_k(x)\,\varphi_k(y) - \sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)$$

$$= \sum_{j=0}^{N} \varphi_{2j}(x)\,\varphi_{2j}(y) - \sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y).$$

It follows that

$$S_N(x,y) = \frac{1}{2}\left[\sum_{j=0}^{2N} \varphi_j(x)\,\varphi_j(y) - \sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)\right].$$

We redefine

$$S_N(x,y) = \sum_{n=0}^{2N} \varphi_n(x)\,\varphi_n(y) = S_N(y,x) \tag{6.13}$$

so that the top left entry of $K_N(x,y)$ is

$$S_N(x,y) + \sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y).$$

If S_N is the operator with kernel $S_N(x,y)$ then integration by parts gives

$$S_N D f = \int_{\mathbb{R}} S(x,y)\frac{d}{dy}f(y)\,dy = \int_{\mathbb{R}}\left(-\frac{d}{dy}S_N(x,y)\right)f(y)\,dy,$$

so that $-\frac{d}{dy}S_N(x,y)$ is in fact the kernel of $S_N D$. Therefore (6.4) now holds with $K_{4,N}$ being the integral operator with matrix kernel $K_{4,N}(x,y)$ whose (i,j)–entry $K_{4,N}^{(i,j)}(x,y)$ is given by

$$K_{4,N}^{(1,1)}(x,y) = \frac{1}{2}\left[S_N(x,y) + \sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)\right],$$

$$K_{4,N}^{(1,2)}(x,y) = \frac{1}{2}\left[SD_N(x,y) - \frac{d}{dy}\left(\sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)\right)\right],$$

$$K_{4,N}^{(2,1)}(x,y) = \frac{\epsilon}{2}\left[S_N(x,y) + \sqrt{\frac{2N+1}{2}}\,\varphi_{2N}(x)\,\epsilon\,\varphi_{2N+1}(y)\right],$$

$$K_{4,N}^{(2,2)}(x,y) = \frac{1}{2}\left[S_N(x,y) + \sqrt{\frac{2N+1}{2}}\,\epsilon\,\varphi_{2N+1}(x)\,\varphi_{2N}(y)\right].$$

We let $2N + 1 \to N$ so that N is assumed to be odd from now on (this will not matter in the end since we will take $N \to \infty$). Therefore the $K_{4,N}^{(i,j)}(x,y)$ are given by

$$K_{4,N}^{(1,1)}(x,y) = \frac{1}{2}\left[S_N(x,y) + \sqrt{\frac{N}{2}}\, \varphi_{N-1}(x)\epsilon\, \varphi_N(y) \right],$$

$$K_{4,N}^{(1,2)}(x,y) = \frac{1}{2}\left[SD_N(x,y) - \sqrt{\frac{N}{2}}\, \varphi_{N-1}(x)\, \varphi_N(y) \right],$$

$$K_{4,N}^{(2,1)}(x,y) = \frac{\epsilon}{2}\left[S_N(x,y) + \sqrt{\frac{N}{2}}\, \varphi_{N-1}(x)\epsilon\, \varphi_N(y) \right],$$

$$K_{4,N}^{(2,2)}(x,y) = \frac{1}{2}\left[S_N(x,y) + \sqrt{\frac{N}{2}}\, \epsilon\, \varphi_N(x)\, \varphi_{N-1}(y) \right],$$

where

$$S_N(x,y) = \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y).$$

Define

$$\varphi(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_N(x) \qquad \psi(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_{N-1}(x),$$

so that

$$K_{4,N}^{(1,1)}(x,y) = \frac{1}{2}\chi(x)\, [S_N(x,y) + \psi(x)\epsilon\,\varphi(y)]\, \chi(y),$$

$$K_{4,N}^{(1,2)}(x,y) = \frac{1}{2}\chi(x)\, [SD_N(x,y) - \psi(x)\varphi(y)]\, \chi(y),$$

$$K_{4,N}^{(2,1)}(x,y) = \frac{1}{2}\chi(x)\, [\epsilon S_N(x,y) + \epsilon\,\psi(x)\epsilon\,\varphi(y)]\, \chi(y),$$

$$K_{4,N}^{(2,2)}(x,y) = \frac{1}{2}\chi(x)\, [S_N(x,y) + \epsilon\,\varphi(x)\,\psi(y)]\, \chi(y).$$

Notice that

$$\frac{1}{2}\chi\, (S + \psi \otimes \epsilon\,\varphi)\, \chi \doteq K_{4,N}^{(1,1)}(x,y),$$

$$\frac{1}{2}\chi\, (SD - \psi \otimes \varphi)\, \chi \doteq K_{4,N}^{(1,2)}(x,y),$$

$$\frac{1}{2}\chi\, (\epsilon S + \epsilon\,\psi \otimes \epsilon\,\varphi)\, \chi \doteq K_{4,N}^{(2,1)}(x,y),$$

$$\frac{1}{2}\chi\, (S + \epsilon\,\varphi \otimes \epsilon\,\psi)\, \chi \doteq K_{4,N}^{(2,2)}(x,y).$$

Therefore

$$K_{4,N} = \frac{1}{2}\,\chi \left(\begin{array}{cc} S + \psi \otimes \epsilon\,\varphi & SD - \psi \otimes \varphi \\ \epsilon\,S + \epsilon\,\psi \otimes \epsilon\,\varphi & S + \epsilon\,\varphi \otimes \psi \end{array} \right) \chi. \tag{6.14}$$

Note that this is identical to the corresponding operator for $\beta = 4$ obtained by Tracy and Widom in [33], the only difference being that φ, ψ, and hence also S, are redefined to depend on λ. This will affect boundary conditions for the differential equations we will obtain later.

6.3   Edge scaling

6.3.1   Reduction of the determinant

We want to compute the Fredholm determinant (6.4) with $K_{4,N}$ given by (6.14) and $f = \chi_{(t,\infty)}$. This is the determinant of an operator on $L^2(J) \times L^2(J)$. Our first task will be to rewrite the determinant as that of an operator on $L^2(J)$. This part follows exactly the proof in [33]. To begin, note that

$$[S,D] = \varphi \otimes \psi + \psi \otimes \varphi \tag{6.15}$$

so that, using the fact that $D\,\epsilon = \epsilon\,D = I$,

$$
\begin{aligned}
[\epsilon,S] &= \epsilon\,S - S\,\epsilon \\
&= \epsilon\,S\,D\,\epsilon - \epsilon\,D\,S\,\epsilon = \epsilon\,[S,D]\,\epsilon \\
&= \epsilon\,\varphi \otimes \psi\,\epsilon + \epsilon\,\psi \otimes \varphi\,\epsilon \\
&= \epsilon\,\varphi \otimes \epsilon^t \psi + \epsilon\,\psi \otimes \epsilon^t\,\varphi \\
&= -\epsilon\,\varphi \otimes \epsilon\,\psi - \epsilon\,\psi \otimes \epsilon\,\varphi,
\end{aligned}
\tag{6.16}
$$

where the last equality follows from the fact that $\epsilon^t = -\epsilon$. We thus have

$$
\begin{aligned}
D\,(\epsilon\,S + \epsilon\,\psi \otimes \epsilon\,\varphi) &= S + \psi \otimes \epsilon\,\varphi, \\
D\,(\epsilon\,S\,D - \epsilon\,\psi \otimes \varphi) &= S\,D - \psi \otimes \varphi.
\end{aligned}
$$

The expressions on the right side are the top matrix entries in (6.14). Thus the first row of $K_{4,N}$ is, as a vector,

$$D\,(\epsilon\,S + \epsilon\,\psi \otimes \epsilon\,\varphi, \epsilon\,S\,D - \epsilon\,\psi \otimes \varphi).$$

Now (6.16) implies that

$$\epsilon\,S + \epsilon\,\psi \otimes \epsilon\,\varphi = S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi.$$

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Similarly (6.15) gives

$$\epsilon \left[S, D \right] = \epsilon \varphi \otimes \psi + \epsilon \psi \otimes \varphi,$$

so that

$$\epsilon S D - \epsilon \psi \otimes \varphi = \epsilon D S + \epsilon \varphi \otimes \psi = S + \epsilon \varphi \otimes \psi.$$

Using these expressions we can rewrite the first row of $K_{4,N}$ as

$$D \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi, S + \epsilon \varphi \otimes \psi \right).$$

Now use (6.16) to show the second row of $K_{4,N}$ is

$$\left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi, S + \epsilon \varphi \otimes \psi \right).$$

Therefore,

$$
\begin{aligned}
K_{4,N} &= \chi \begin{pmatrix} D \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) & D \left(S + \epsilon \varphi \otimes \psi \right) \\ S \epsilon - \epsilon \varphi \otimes \epsilon \psi & S + \epsilon \varphi \otimes \psi \end{pmatrix} \chi \\
&= \begin{pmatrix} \chi D & 0 \\ 0 & \chi \end{pmatrix} \begin{pmatrix} \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \chi & \left(S + \epsilon \varphi \otimes \psi \right) \chi \\ \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \chi & \left(S + \epsilon \varphi \otimes \psi \right) \chi \end{pmatrix}.
\end{aligned}
$$

Since $K_{4,N}$ is of the form $A B$, we can use 4.2 and deduce that $D_{4,N}(s, \lambda)$ is unchanged if instead we take $K_{4,N}$ to be

$$
\begin{aligned}
K_{4,N} &= \begin{pmatrix} \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \chi & \left(S + \epsilon \varphi \otimes \psi \right) \chi \\ \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \chi & \left(S + \epsilon \varphi \otimes \psi \right) \chi \end{pmatrix} \begin{pmatrix} \chi D & 0 \\ 0 & \chi \end{pmatrix} \\
&= \begin{pmatrix} \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \chi D & \left(S + \epsilon \varphi \otimes \psi \right) \chi \\ \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \chi D & \left(S + \epsilon \varphi \otimes \psi \right) \chi \end{pmatrix}.
\end{aligned}
$$

Therefore

$$D_{4,N}(s, \lambda) = \det \begin{pmatrix} I - \frac{1}{2} \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \lambda \chi D & -\frac{1}{2} \left(S + \epsilon \varphi \otimes \psi \right) \lambda \chi \\ -\frac{1}{2} \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \lambda \chi D & I - \frac{1}{2} \left(S + \epsilon \varphi \otimes \psi \right) \lambda \chi \end{pmatrix}.$$

$$(6.17)$$

Now we perform row and column operations on the matrix to simplify it, which do not change the Fredholm determinant. Justification of these operations is given in [33]. We start by subtracting row 1 from row 2 to get

$$\begin{pmatrix} I - \frac{1}{2} \left(S \epsilon - \epsilon \varphi \otimes \epsilon \psi \right) \lambda \chi D & -\frac{1}{2} \left(S + \epsilon \varphi \otimes \psi \right) \lambda \chi \\ -I & I \end{pmatrix}.$$

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Next, adding column 2 to column 1 yields

$$
\begin{pmatrix}
I - \frac{1}{2}\left(S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi\right)\lambda\,\chi & D - \frac{1}{2}\left(S + \epsilon\,\varphi \otimes \psi\right)\lambda\,\chi & -\frac{1}{2}\left(S + \epsilon\,\varphi \otimes \psi\right)\lambda\,\chi \\
0 & & I
\end{pmatrix}.
$$

Thus the determinant we want equals the determinant of

$$
I - \frac{1}{2}\left(S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi\right)\lambda\,\chi\, D - \frac{1}{2}\left(S + \epsilon\,\varphi \otimes \psi\right)\lambda\,\chi \qquad (6.18)
$$

So we have reduced the problem from the computation of the Fredholm determinant of an operator on $L^2(J) \times L^2(J)$, to that of an operator on $L^2(J)$.

6.3.2 Differential equations

Next we want to write the operator in (6.18) in the form

$$
(I - K_{2,N})\left(I - \sum_{i=1}^{L}\alpha_i \otimes \beta_i\right), \qquad (6.19)
$$

where the α_i and β_i are functions in $L^2(J)$. In other words, we want to rewrite the determinant for the GSE case as a finite dimensional perturbation of the corresponding GUE determinant. The Fredholm determinant of the product is then the product of the determinants. The limiting form for the GUE part is already known, and we can just focus on finding a limiting form for the determinant of the finite dimensional piece. It is here that the proof must be modified from that in [33]. A little rearrangement of (6.18) yields (recall $\epsilon^t = -\epsilon$)

$$
I - \frac{\lambda}{2}S\chi - \frac{\lambda}{2}S\,\epsilon\,\chi\,D - \frac{\lambda}{2}\epsilon\,\varphi \otimes \chi\,\psi - \frac{\lambda}{2}\epsilon\,\varphi \otimes \psi\,\epsilon\,\chi\,D.
$$

Writing $\epsilon\left[\chi, D\right] + \chi$ for $\epsilon\,\chi\,D$ and simplifying gives

$$
I - \lambda\,S\chi - \lambda\,\epsilon\,\varphi \otimes \psi\,\chi - \frac{\lambda}{2}S\,\epsilon\left[\chi, D\right] - \frac{\lambda}{2}\epsilon\,\varphi \otimes \psi\,\epsilon\left[\chi, D\right].
$$

Let $\sqrt{\lambda}\,\varphi \to \varphi$, and $\sqrt{\lambda}\,\psi \to \psi$ so that $\lambda\,S \to S$ and (6.18) goes to

$$
I - S\chi - \epsilon\,\varphi \otimes \psi\,\chi - \frac{1}{2}S\,\epsilon\left[\chi, D\right] - \frac{1}{2}\epsilon\,\varphi \otimes \psi\,\epsilon\left[\chi, D\right].
$$

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Now we define $R := (I - S\chi)^{-1} S\chi = (I - S\chi)^{-1} - I$ (the resolvent operator of $S\chi$), whose kernel we denote by $R(x, y)$, and $Q_\epsilon := (I - S\chi)^{-1} \epsilon\varphi$. Then (6.18) factors into

$$A = (I - S\chi) B.$$

where B is

$$I - Q_\epsilon \otimes \chi\psi - \frac{1}{2}(I + R) S\epsilon\,[\chi, D] - \frac{1}{2}(Q_\epsilon \otimes \psi)\epsilon\,[\chi, D]$$

Hence

$$D_{4,N}(t, \lambda) = D_{2,N}(t, \lambda)\,\det(B).$$

In order to find $\det(B)$ we use the identity

$$\epsilon\,[\chi, D] = \sum_{k=1}^{2m}(-1)^k \epsilon_k \otimes \delta_k, \tag{6.20}$$

where ϵ_k and δ_k are the functions $\epsilon(x - a_k)$ and $\delta(x - a_k)$ respectively, and the a_k are the endpoints of the (disjoint) intervals considered, $J = \cup_{k=1}^m (a_{2k-1}, a_{2k})$. In our case $m = 1$ and $a_1 = t$, $a_2 = \infty$. We also make use of the fact that

$$a \otimes b \cdot c \otimes d = (b, c) \cdot a \otimes d \tag{6.21}$$

where $(\,.\,,.\,)$ is the usual L^2–inner product. Therefore

$$(Q_\epsilon \otimes \psi)\epsilon\,[\chi, D] = \sum_{k=1}^{2}(-1)^k Q_\epsilon \otimes \psi \cdot \epsilon_k \otimes \delta_k$$

$$= \sum_{k=1}^{2}(-1)^k (\psi, \epsilon_k)\,Q_\epsilon \otimes \delta_k.$$

It follows that

$$\frac{D_{4,N}(t, \lambda)}{D_{2,N}(t, \lambda)} \tag{6.22}$$

is the determinant of

$$I - Q_\epsilon \otimes \chi\psi - \frac{1}{2}\sum_{k=1}^{2}(-1)^k\,[(S + R\,S)\,\epsilon_k + (\psi, \epsilon_k)\,Q_\epsilon] \otimes \delta_k. \tag{6.23}$$

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We now specialize to the case of one interval $J = (t, \infty)$, so $m = 1$, $a_1 = t$ and $a_2 = \infty$. We write $\epsilon_t = \epsilon_1$, and $\epsilon_\infty = \epsilon_2$, and similarly for δ_k. Writing out the terms in the summation and using the fact that

$$\epsilon_\infty = -\frac{1}{2}, \tag{6.24}$$

yields

$$I - Q_\epsilon \otimes \chi \psi + \frac{1}{2} \left[(S + RS)\, \epsilon_t + (\,\psi\,,\epsilon_t\,)\, Q_\epsilon \right] \otimes \delta_t + \frac{1}{4} \left[(S + RS)\, 1 + (\,\psi\,,1\,)\, Q_\epsilon \right] \otimes \delta_\infty \tag{6.25}$$

Now we can use the formula

$$\det \left(I - \sum_{i=1}^{L} \alpha_i \otimes \beta_i \right) = \det \left(\delta_{jk} - (\,\alpha_j\,,\beta_k\,) \right)_{1 \le j,k \le L} \tag{6.26}$$

In order to simplify the notation in preparation for the computation of the various inner products, define

$$\begin{aligned}
Q(x, \lambda, t) &:= (I - S\chi)^{-1}\, \varphi, & P(x, \lambda, t) &:= (I - S\chi)^{-1}\, \psi, \\
Q_\epsilon(x, \lambda, t) &:= (I - S\chi)^{-1}\, \epsilon\, \varphi, & P_\epsilon(x, \lambda, t) &:= (I - S\chi)^{-1}\, \epsilon\, \psi, \tag{6.27}
\end{aligned}$$

$$\begin{aligned}
q_N &:= Q(t, \lambda, t), & p_N &:= P(t, \lambda, t), \\
q_\epsilon &:= Q_\epsilon(t, \lambda, t), & p_\epsilon &:= P_\epsilon(t, \lambda, t), \\
u_\epsilon &:= (\,Q\,,\chi\,\epsilon\,\varphi\,) = (\,Q_\epsilon\,,\chi\,\varphi\,), & v_\epsilon &:= (\,Q\,,\chi\,\epsilon\,\psi\,) = (\,P_\epsilon\,,\chi\,\psi\,), \\
\tilde{v}_\epsilon &:= (\,P\,,\chi\,\epsilon\,\varphi\,) = (\,Q_\epsilon\,,\chi\,\varphi\,), & w_\epsilon &:= (\,P\,,\chi\,\epsilon\,\psi\,) = (\,P_\epsilon\,,\chi\,\psi\,), \tag{6.28}
\end{aligned}$$

$$\mathcal{P}_4 := \int_{\mathbb{R}} \epsilon_t(x)\, P(x, t)\, d\,x, \quad \mathcal{Q}_4 := \int_{\mathbb{R}} \epsilon_t(x)\, Q(x, t)\, d\,x, \quad \mathcal{R}_4 := \int_{\mathbb{R}} \epsilon_t(x)\, R(x, t)\, d\,x, \tag{6.29}$$

where we remind the reader that ϵ_t stands for the function $\epsilon(x - t)$. Note that all quantities in (6.28) and (6.29) are functions of t and λ alone. Furthermore, let

$$c_\varphi = \epsilon\, \varphi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi(x)\, d\,x, \qquad c_\psi = \epsilon\, \psi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(x)\, d\,x. \tag{6.30}$$

Recall from the previous section that when $\beta = 4$ we take N to be odd. It follows that φ and ψ are odd and even functions respectively. Thus when $\beta = 4$, $c_\varphi = 0$ while computation using known integrals for the Hermite polynomials gives

$$c_\psi = (\pi N)^{1/4} 2^{-3/4 - N/2} \frac{(N!)^{1/2}}{(N/2)!} \sqrt{\lambda}. \tag{6.31}$$

Hence computation yields

$$\lim_{N \to \infty} c_\psi = \sqrt{\frac{\lambda}{2}}. \tag{6.32}$$

At $t = \infty$,

$$u_\epsilon(\infty) = 0, \quad q_\epsilon(\infty) = c_\varphi \tag{6.33}$$

$$\mathcal{P}_4(\infty) = -c_\psi, \quad \mathcal{Q}_4(\infty) = -c_\varphi, \quad \mathcal{R}_4(\infty) = 0. \tag{6.34}$$

In (6.26), $L = 3$ and if we denote $a_4 = (\psi, \epsilon_t)$, then we have explicitly

$$\alpha_1 = Q_\epsilon, \quad \alpha_2 = -\frac{1}{2} \left[(S + R S) \epsilon_t + a_4 Q_\epsilon \right], \quad \alpha_3 = -\frac{1}{4} \left[(S + R S) 1 + (\psi, 1) Q_\epsilon \right],$$

$$\beta_1 = \chi\psi, \quad \beta_2 = \delta_t, \quad \beta_3 = \delta_\infty.$$

However notice that

$$((S + R S) \epsilon_t, \delta_\infty) = (\epsilon_t, \delta_\infty) = 0, \quad ((S + R S) 1, \delta_\infty) = (1, R_\infty) = 0 \tag{6.35}$$

and $(Q_\epsilon, \delta_\infty) = c_\varphi = 0$. Therefore the terms involving $\beta_3 = \delta_\infty$ are all 0 and we can discard them reducing our computation to that of a 2×2 determinant instead with

$$\alpha_1 = Q_\epsilon, \quad \alpha_2 = -\frac{1}{2} \left[(S + R S) \epsilon_t + a_4 Q_\epsilon \right], \quad \beta_1 = \chi\psi, \quad \beta_2 = \delta_t. \tag{6.36}$$

Hence

$$(\alpha_1, \beta_1) = \tilde{v}_\epsilon, \quad (\alpha_1, \beta_2) = q_\epsilon, \tag{6.37}$$

$$(\alpha_2, \beta_1) = -\frac{1}{2} (\mathcal{P}_4 - a_4 + a_4 \tilde{v}_\epsilon), \tag{6.38}$$

$$(\alpha_2, \beta_2) = -\frac{1}{2} (\mathcal{R}_4 + a_4 q_\epsilon). \tag{6.39}$$

We want the limit of the determinant

$$\det\left(\delta_{jk} - (\,\alpha_j\,,\beta_k\,)\right)_{1\leq j,k\leq 2}, \qquad (6.40)$$

as $N \to \infty$. In order to get our hands on the limits of the individual terms involved in the determinant, we will find differential equations for them first as in [33]. Adding $a_4/2$ times row 1 to row 2 shows that a_4 falls out of the determinant, so we will not need to find differential equations for it. Thus our determinant is now

$$\det\begin{pmatrix} 1 - \tilde{v}_\epsilon & -q_\epsilon \\ \frac{1}{2}\mathcal{P}_4 & 1 + \frac{1}{2}\mathcal{R}_4 \end{pmatrix}. \qquad (6.41)$$

Proceeding as in [33] we find the following differential equations

$$\frac{d}{dt}u_\epsilon = -q_N\, q_\epsilon, \qquad\qquad \frac{d}{dt}q_\epsilon = q_N - q_N\,\tilde{v}_\epsilon - p_N\,u_\epsilon, \qquad (6.42)$$

$$\frac{d}{dt}\mathcal{Q}_4 = -q_N\,(\mathcal{R}_4 + 1)\,, \qquad \frac{d}{dt}\mathcal{P}_4 = -p_N\,(\mathcal{R}_4 + 1)\,, \qquad (6.43)$$

$$\frac{d}{dt}\mathcal{R}_4 = -p_N\,\mathcal{Q}_4 - q_N\,\mathcal{P}_4. \qquad (6.44)$$

Now we change variable from t to s where $t = \tau(s) = \sqrt{2N} + s/(\sqrt{2}\,N^{1/6})$ and take the limit $N \to \infty$, denoting the limits of q_ϵ, $\mathcal{P}_4$, $\mathcal{Q}_4$, $\mathcal{R}_4$, and the common limit of u_ϵ and $\tilde{v}_\epsilon$ respectively by $\overline{q}$, $\overline{\mathcal{P}}_4$, $\overline{\mathcal{Q}}_4$, $\overline{\mathcal{R}}_4$ and $\overline{u}$. Also $\overline{\mathcal{P}}_4$ and $\overline{\mathcal{Q}}_4$ differ by a constant, namely $\overline{\mathcal{Q}}_4 = \overline{\mathcal{P}}_4 + \sqrt{2}/2$. These limits hold uniformly for bounded s so we can interchange $\lim_{N\to\infty}$ and $\frac{d}{ds}$. Also $\lim_{N\to\infty} N^{-1/6}q_N = \lim_{N\to\infty} N^{-1/6}p_N = q$, where q is as in (3.10). We obtain the systems

$$\frac{d}{ds}\overline{u} = -\frac{1}{\sqrt{2}}\,q\,\overline{q}, \qquad \frac{d}{ds}\overline{q} = \frac{1}{\sqrt{2}}\,q\,(1 - 2\,\overline{u})\,, \qquad (6.45)$$

$$\frac{d}{ds}\overline{\mathcal{P}}_4 = -\frac{1}{\sqrt{2}}\,q\,(\overline{\mathcal{R}}_4 + 1)\,, \qquad \frac{d}{ds}\overline{\mathcal{R}}_4 = -\frac{1}{\sqrt{2}}\,q\left(2\,\overline{\mathcal{P}}_4 + \sqrt{\frac{\lambda}{2}}\right), \qquad (6.46)$$

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The change of variables $s \rightarrow \mu = \int_s^\infty q(x, \lambda) \, dx$ transforms these systems into constant coefficient ordinary differential equations

$$\frac{d}{d\mu} \overline{u} = \frac{1}{\sqrt{2}} \, \overline{q}, \qquad \frac{d}{d\mu} \overline{q} = -\frac{1}{\sqrt{2}} \, (1 - 2\,\overline{u}), \tag{6.47}$$

$$\frac{d}{d\mu} \overline{\mathcal{P}}_4 = \frac{1}{\sqrt{2}} \, (\overline{\mathcal{R}}_4 + 1), \qquad \frac{d}{d\mu} \overline{\mathcal{R}}_4 = \frac{1}{\sqrt{2}} \left(2\overline{\mathcal{P}}_4 + \sqrt{\frac{\lambda}{2}} \right). \tag{6.48}$$

Since $\lim_{s \to \infty} \mu = 0$, corresponding to the boundary values at $t = \infty$ which we found earlier for $\mathcal{P}_4, \mathcal{R}_4$, we now have initial values at $\mu = 0$. Therefore

$$\overline{u}(\mu = 0) = \overline{q}(\mu = 0) = 0, \tag{6.49}$$

$$\overline{\mathcal{P}}_4(\mu = 0) = -\sqrt{\frac{\lambda}{2}}, \qquad \overline{\mathcal{R}}_4(\mu = 0) = 0. \tag{6.50}$$

We use this to solve the systems and get

$$\overline{q} = \frac{1}{2\sqrt{2}} \left(e^{-\mu} - e^{\mu} \right) \tag{6.51}$$

$$\overline{u} = \frac{1}{2} \left(1 - \frac{1}{2} e^{\mu} - \frac{1}{2} e^{-\mu} \right) \tag{6.52}$$

$$\overline{\mathcal{P}}_4 = \frac{1}{2\sqrt{2}} \left(\frac{2 - \sqrt{\lambda}}{2} e^{\mu} - \frac{2 + \sqrt{\lambda}}{2} e^{-\mu} - \sqrt{\lambda} \right) \tag{6.53}$$

$$\overline{\mathcal{R}}_4 = \frac{2 - \sqrt{\lambda}}{4} e^{\mu} + \frac{2 + \sqrt{\lambda}}{4} e^{-\mu} - 1 \tag{6.54}$$

Substituting these expressions into the determinant gives (3.13), namely

$$D_4(s, \lambda) = D_2(s, \lambda) \cosh^2 \left(\frac{\mu(s, \lambda)}{2} \right), \tag{6.55}$$

where $D_\beta = \lim_{N \to \infty} D_{\beta,N}$. Note that even though there are λ–terms in (6.53) and (6.54), these do not appear in the final result (6.55), making it similar to the GUE case where the main conceptual difference between the $m = 1$ (largest eigenvalue) case and the general m is the dependence of the function q on λ. The right hand side of the above formula clearly reduces to the $\beta = 4$ Tracy-Widom distribution when we set $\lambda = 1$. Note

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that where we have $D_4(s, \lambda)$ above, Tracy and Widom (and hence many RMT references) write $D_4(s/\sqrt{2}, \lambda)$ instead. Tracy and Widom applied the change of variable $s \to s/\sqrt{2}$ in their derivation in [33] so as to agree with Mehta's form of the $\beta = 4$ joint eigenvalue density,[6] which has $-2x^2$ in the exponential in the weight function, instead of $-x^2$ in our case. To switch back to the other convention, one just needs to substitute in the argument $s/\sqrt{2}$ for s everywhere in our results. At this point this is just a cosmetic discrepancy, and it does not change anything in our derivations since all the differentiations are done with respect to λ anyway. It <u>does</u> change conventions for rescaling data while doing numerical work though.

7 The Distribution of the mth Largest Eigenvalue in the GOE

7.1 The distribution function as a Fredholm determinant

The GOE corresponds case corresponds to the specialization $\beta = 1$ in (3.1) so that

$$
G_{1,N}(t, \lambda) = C_1^{(N)} \int \cdots \int_{x_i \in \mathbb{R}} \prod_{j<k} |x_j - x_k| \prod_j^N w(x_j) \prod_j^N (1 + f(x_j)) \, d\,x_1 \cdots d\,x_N
$$

(7.1)

where $w(x) = \exp\left(-x^2\right)$, $f(x) = -\lambda \chi_J(x)$, and $C_1^{(N)}$ depends only on N. As in the GSE case, we will lump into $C_1^{(N)}$ any constants depending only on N that appear in the derivation. A simple argument at the end will show that the final constant is 1. These calculations more or less faithfully follow and expand on [34]. We want to use (4.2), which requires an ordered space. Note that the above integrand is symmetric under permutations, so the integral is $n!$ times the same integral over ordered pairs $x_1 \leq \ldots \leq x_N$. So we can rewrite 7.1 as

$$
(N!) \int \cdots \int_{x_1 \leq \ldots \leq x_N \in \mathbb{R}} \prod_{j<k} (x_k - x_j) \prod_{i=1}^N w(x_k) \prod_{i=1}^N (1 + f(x_k)) \, d\,x_1 \cdots d\,x_N,
$$

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where we can remove the absolute values since the ordering insures that $(x_j - x_i) \geq 0$ for $i < j$. Recall that the Vandermonde determinant is

$$\Delta_N(x) = \det(x_k^{j-1})_{1 \leq j,k \leq N} = (-1)^{\frac{N(N-1)}{2}} \prod_{j<k} (x_j - x_k).$$

Therefore what we have inside the integrand above is, up to sign

$$\det(x_k^{j-1} \, w(x_k) \, (1 + f(x_k)))_{1 \leq j,k \leq N}.$$

Note that the sign depends only on N. Now we can use (4.2) with

$$\varphi_j(x) = x^{j-1} \, w(x) \, (1 + f(x)).$$

In using (4.2) we square both sides so that the right hand side is now a determinant instead of a Pfaffian. Therefore $G_{1,N}^2(t,\lambda)$ equals

$$C_1^{(N)} \det \left(\int \int \operatorname{sgn}(x-y) x^{j-1} \, y^{k-1} \, (1 + f(x)) \, w(x) \, w(y) \, dx \, dy \right)_{1 \leq j,k \leq N}.$$

Shifting indices, we can write it as

$$C_1^{(N)} \det \left(\int \int \operatorname{sgn}(x-y) x^j \, y^k \, (1 + f(x)) \, w(x) \, w(y) \, dx \, dy \right)_{1 \leq j,k \leq N-1} \tag{7.2}$$

where $C_1^{(N)}$ is a constant depending only on N, and is such that the right side is 1 if $f \equiv 0$. Indeed this would correspond to the probability that $\lambda_p^{GOE(N)} < \infty$, or equivalently to the case where the excluded set J is empty. We can replace x^j and y^k by any arbitrary polynomials $p_j(x)$ and $p_k(x)$, of degree j and k respectively, which are obtained by row operations on the matrix. Indeed such operations would not change the determinant. We also replace $\operatorname{sgn}(x-y)$ by $\epsilon(x-y) = \frac{1}{2} \operatorname{sgn}(x-y)$ which just produces a factor of 2 that we absorb in $C_1^{(N)}$. Thus $G_{1,N}^2(t,\lambda)$ now equals

$$C_1^{(N)} \det \left(\int \int \epsilon(x-y) \, p_j(x) \, p_k(y) \, (1 + f(x)) \, (1 + f(y)) \, w(x) \, w(y) \, dx \, dy \right)_{0 \leq j,k \leq N-1}. \tag{7.3}$$

Let $\psi_j(x) = p_j(x) \, w(x)$ so the above integral becomes

$$C_1^{(N)} \det \left(\int \int \epsilon(x-y) \, \psi_j(x) \, \psi_k(y) \, (1 + f(x) + f(y) + f(x) \, f(y)) \, dx \, dy \right)_{0 \leq j,k \leq N-1}. \tag{7.4}$$

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Partially multiplying out the term we obtain

$$
C_1^{(N)} \det \left(\int \int \epsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, d\,x\, d\,y \right.
$$

$$
\left. + \int \int \epsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, (f(x) + f(y) + f(x)\, f(y))\, d\,x\, d\,y \right)_{0 \le j,k \le N-1}.
$$

Define

$$
M = \left(\int \int \epsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, d\,x\, d\,y \right)_{0 \le j,k \le N-1}, \tag{7.5}
$$

so that $G_{1,N}^2(t, \lambda)$ is now

$$
C_1^{(N)} \det \left(M + \int \int \epsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, (f(x) + f(y) + f(x)\, f(y))\, d\,x\, d\,y \right)_{0 \le j,k \le N-1}.
$$

Let ϵ be the operator defined in (6.2). We can use operator notation to simplify the expression for $G_{1,N}^2(t, \lambda)$ a great deal by rewriting the double integrals as single integrals. Indeed

$$
\int \int \epsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, f(x)\, d\,x\, d\,y = \int f(x)\, \psi_j(x) \int \epsilon(x - y)\, \psi_k(y)\, d\,y\, d\,x
$$

$$
= \int f\, \psi_j\, \epsilon\, \psi_k\, d\,x.
$$

Similarly,

$$
\int \int \epsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, f(y)\, d\,x\, d\,y = -\int \int \epsilon(y - x)\, \psi_j(x)\, \psi_k(y)\, f(y)\, d\,x\, d\,y
$$

$$
= -\int f(y)\, \psi_k(y) \int \epsilon(y - x)\, \psi_j(x)\, d\,x\, d\,y
$$

$$
= -\int f(x)\, \psi_k(x) \int \epsilon(x - y)\, \psi_j(y)\, d\,y\, d\,x
$$

$$
= -\int f\, \psi_k\, \epsilon\, \psi_j\, d\,x.
$$

Finally,

$$\int\int \epsilon(x-y)\,\psi_j(x)\,\psi_k(y)\,f(x)\,f(y)\,dx\,dy$$

$$= -\int\int \epsilon(y-x)\,\psi_j(x)\,\psi_k(y)\,f(x)\,f(y)\,dx\,dy$$

$$= -\int f(y)\,\psi_k(y)\int \epsilon(y-x)\,f(x)\,\psi_j(x)\,dx\,dy \quad (7.6)$$

$$= -\int f(x)\,\psi_k(x)\int \epsilon(x-y)\,f(y)\,\psi_j(y)\,dy\,dx$$

$$= -\int f\,\psi_k\,\epsilon\,(f\,\psi_j)\,dx.$$

It follows that

$$G^2_{1,N}(t,\lambda) = C_1^{(N)}\,\det\left(M + \int [f\,\psi_j\,\epsilon\,\psi_k - f\,\psi_k\,\epsilon\,\psi_j - f\,\psi_k\,\epsilon\,(f\,\psi_j)]\,dx\right)_{0\le j,k\le N-1}.$$
$$(7.7)$$

If we let $M^{-1} = (\mu_{jk})_{0\le j,k\le N-1}$, and factor $\det(M)$ out, then $G^2_{1,N}(t,\lambda)$ equals

$$C_1^{(N)}\,\det(M)\,\det\Bigg(I +$$

$$M^{-1}\cdot\left(\int [f\,\psi_j\,\epsilon\,\psi_k - f\,\psi_k\,\epsilon\,\psi_j - f\,\psi_k\,\epsilon\,(f\,\psi_j)]\,dx\right)_{0\le j,k\le N-1}\Bigg)_{0\le j,k\le N-1}$$
$$(7.8)$$

where the dot denotes matrix multiplication of M^{-1} and the matrix with the integral as its (j,k)–entry. define $\eta_j = \sum_j \mu_{jk}\,\psi_k$ and use it to simplify the result of carrying out the matrix multiplication. From (7.5) it follows that $\det(M)$ depends only on N we lump it into $C_1^{(N)}$. Thus $G^2_{1,N}(t,\lambda)$ equals

$$C_1^{(N)}\,\det\left(I + \left(\int [f\,\eta_j\,\epsilon\,\psi_k - f\,\psi_k\,\epsilon\,\eta_j - f\,\psi_k\,\epsilon\,(f\,\eta_j)]\,dx\right)_{0\le j,k\le N-1}\right)_{0\le j,k\le N-1}.$$
$$(7.9)$$

Recall our remark at the very beginning of the section that if $f\equiv 0$ then the integral we started with evaluates to 1 so that

$$C_1^{(N)}\,\det(I) = C_1^{(N)}, \quad (7.10)$$

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which implies that $C_1^{(N)} = 1$. Now $G_{1,N}^2(t, \lambda)$ is of the form $\det(I + AB)$ where $A : L^2(J) \times L^2(J) \to \mathbb{C}^N$ is a $N \times 2$ matrix

$$
A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix},
$$

whose j^{th} row is given by

$$
A_j = A_j(x) = \left(-f \, \epsilon \, \eta_j - f \, \epsilon \, (f \, \eta_j) \qquad f \, \eta_j \right).
$$

Therefore, if

$$
g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L^2(J) \times L^2(J),
$$

then $A g$ is a column vector whose j^{th} row is $(A_j , g)_{L^2 \times L^2}$

$$
(A g)_j = \int \left[-f \, \epsilon \, \eta_j - f \, \epsilon \, (f \, \eta_j) \right] g_1 \, d \, x + \int f \, \eta_j \, g_2 \, d \, x.
$$

Similarly, $B : \mathbb{C}^N \to L^2(J) \times L^2(J)$ is a $2 \times N$ matrix

$$
B = \begin{pmatrix} B_1 & B_2 & \ldots & B_N \end{pmatrix},
$$

whose j^{th} column is given by

$$
B_j = B_j(x) = \begin{pmatrix} \psi_j \\ \epsilon \, \psi_j \end{pmatrix}.
$$

Thus if

$$
h = \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} \in \mathbb{C}^N,
$$

then $B h$ is the column vector of $L^2(J) \times L^2(J)$ given by

$$
B h = \begin{pmatrix} \sum_j h_i \, \psi_j \\ \sum_j h_i \, \epsilon \, \psi_j \end{pmatrix}.
$$

Clearly $A\,B : \mathbb{C}^N \to \mathbb{C}^N$ and $B\,A : L^2(J) \times L^2(J) \to L^2(J) \times L^2(J)$ with kernel

$$
\begin{pmatrix}
-\sum_j \psi_j \otimes f\,\epsilon\,\eta_j - \sum_j \psi_j \otimes f\,\epsilon\,(f\,\eta_j) & \sum_j \psi_j \otimes f\,\eta_j \\
-\sum_j \epsilon\,\psi_j \otimes f\,\epsilon\,\eta_j - \sum_j \epsilon\,\psi_j \otimes f\,\epsilon\,(f\,\eta_j) & \sum_j \epsilon\,\psi_j \otimes f\,\eta_j
\end{pmatrix}.
$$

Hence $I + BA$ has kernel

$$
\begin{pmatrix}
I - \sum_j \psi_j \otimes f\,\epsilon\,\eta_j - \sum_j \psi_j \otimes f\,\epsilon\,(f\,\eta_j) & \sum_j \psi_j \otimes f\,\eta_j \\
-\sum_j \epsilon\,\psi_j \otimes f\,\epsilon\,\eta_j - \sum_j \epsilon\,\psi_j \otimes f\,\epsilon\,(f\,\eta_j) & I + \sum_j \epsilon\,\psi_j \otimes f\,\eta_j
\end{pmatrix},
$$

which can be written as

$$
\begin{pmatrix}
I - \sum_j \psi_j \otimes f\,\epsilon\,\eta_j & \sum_j \psi_j \otimes f\,\eta_j \\
-\sum_j \epsilon\,\psi_j \otimes f\,\epsilon\,\eta_j - \epsilon\,f & I + \sum_j \epsilon\,\psi_j \otimes f\,\eta_j
\end{pmatrix} \cdot
\begin{pmatrix}
I & 0 \\
\epsilon\,f & I
\end{pmatrix}.
$$

Since we are taking the determinant of this operator expression, and the determinant of the second term is just 1, we can drop it. Therefore

$$
\begin{aligned}
G_{1,N}^2(t,\lambda) &= \det \begin{pmatrix}
I - \sum_j \psi_j \otimes f\,\epsilon\,\eta_j & \sum_j \psi_j \otimes f\,\eta_j \\
-\sum_j \epsilon\,\psi_j \otimes f\,\epsilon\,\eta_j - \epsilon\,f & I + \sum_j \epsilon\,\psi_j \otimes f\,\eta_j
\end{pmatrix} \\
&= \det(I + K_{1,N}\,f),
\end{aligned}
$$

where

$$
\begin{aligned}
K_{1,N} &= \begin{pmatrix}
-\sum_j \psi_j \otimes \epsilon\,\eta_j & \sum_j \psi_j \otimes \eta_j \\
-\sum_j \epsilon\,\psi_j \otimes \epsilon\,\eta_j - \epsilon & \sum_j \epsilon\,\psi_j \otimes \eta_j
\end{pmatrix} \\
&= \begin{pmatrix}
-\sum_{j,k} \psi_j \otimes \mu_{jk}\,\epsilon\,\psi_k & \sum_{j,k} \psi_j \otimes \mu_{jk}\,\psi_k \\
-\sum_{j,k} \epsilon\,\psi_j \otimes \mu_{jk}\,\epsilon\,\psi_k - \epsilon & \sum_{j,k} \epsilon\,\psi_j \otimes \mu_{jk}\,\psi_k
\end{pmatrix}
\end{aligned}
$$

and $K_{1,N}$ has matrix kernel

$$
K_{1,N}(x,y) = \begin{pmatrix}
-\sum_{j,k} \psi_j(x)\,\mu_{jk}\,\epsilon\,\psi_k(y) & \sum_{j,k} \psi_j(x)\,\mu_{jk}\,\psi_k(y) \\
-\sum_{j,k} \epsilon\,\psi_j(x)\,\mu_{jk}\,\epsilon\,\psi_k(y) - \epsilon(x-y) & \sum_{j,k} \epsilon\,\psi_j(x)\,\mu_{jk}\,\psi_k(y)
\end{pmatrix}.
$$

We define

$$
S_N(x,y) = -\sum_{j,k=0}^{N-1} \psi_j(x)\mu_{jk}\epsilon\,\psi_k(y).
$$

Since M is antisymmetric,

$$
S_N(y,x) = -\sum_{j,k=0}^{N-1} \psi_j(y)\mu_{jk}\epsilon\,\psi_k(x) = \sum_{j,k=0}^{N-1} \psi_j(y)\mu_{kj}\epsilon\,\psi_k(x) = \sum_{j,k=0}^{N-1} \epsilon\,\psi_j(x)\mu_{jk}\psi_k(y).
$$

Note that

$$\epsilon\, S_N(x,y) = \sum_{j,k=0}^{N-1} \epsilon\, \psi_j(x)\, \mu_{jk}\, \epsilon\, \psi_k(y),$$

whereas

$$-\frac{d}{d\,y}\, S_N(x,y) = \sum_{j,k=0}^{N-1} \psi_j(x)\mu_{jk}\psi_k(y).$$

So we can now write succinctly

$$K_{1,N}(x,y) = \begin{pmatrix} S_N(x,y) & -\frac{d}{dy} S_N(x,y) \\ \epsilon\, S_N(x,y) - \epsilon & S_N(y,x) \end{pmatrix} \tag{7.11}$$

So we have shown that

$$G_{1,N}(t,\lambda) = \sqrt{D_{1,N}(t,\lambda)} \tag{7.12}$$

where

$$D_{1,N}(t,\lambda) = \det\left(I + K_{1,N}\, f\right)$$

where $K_{1,N}$ is the integral operator with matrix kernel $K_{1,N}(x,y)$ given in (7.11).

7.2 Gaussian specialization

We specialize the results above to the case of a Gaussian weight function

$$w(x) = \exp\left(-x^2/2\right) \tag{7.13}$$

and indicator function

$$f(x) = -\lambda\, \chi_{_J} \qquad J = (t,\infty)$$

Note that this does not agree with the weight function in (3.1). However it is a necessary choice if we want the technical convenience of working with exactly the same orthogonal polynomials (the Hermite functions) as in the $\beta = 2, 4$ cases. In turn the Painlevé function in the limiting distribution will be unchanged. The discrepancy is resolved by the choice of standard deviation. Namely here the standard deviation on the diagonal matrix elements is taken to be 1, corresponding to the weight function (7.13). In the $\beta = 2, 4$ cases

the standard deviation on the diagonal matrix elements is $1/\sqrt{2}$, giving the weight function (6.6). Now we again want the matrix

$$M = \left(\int \int \epsilon(x - y)\,\psi_j(x)\,\psi_k(y)\,d\,x\,d\,y \right)_{0 \le j,k \le N-1} = \left(\int \psi_j(x)\,\epsilon\,\psi_k(x)\,d\,x \right)_{0 \le j,k \le N-1}$$

to be the direct sum of $\frac{N}{2}$ copies of

$$Z = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

so that the formulas are the simplest possible, since then μ_{jk} can only be 0 or ± 1. In that case M would be skew–symmetric so that $M^{-1} = -M$. In terms of the integrals defining the entries of M this means that we would like to have

$$\int \psi_{2m}(x)\,\epsilon\,\psi_{2n+1}(x)\,d\,x = \delta_{m,n},$$

$$\int \psi_{2m+1}(x)\,\epsilon\,\psi_{2n}(x)\,d\,x = -\delta_{m,n},$$

and otherwise

$$\int \psi_j(x)\,\frac{d}{d\,x}\,\psi_k(x)\,d\,x = 0.$$

It is easier to treat this last case if we replace it with three non-exclusive conditions

$$\int \psi_{2m}(x)\,\epsilon\,\psi_{2n}(x)d\,x = 0,$$

$$\int \psi_{2m+1}(x)\,\epsilon\,\psi_{2n+1}(x)\,d\,x = 0$$

(so when the parity is the same for j, k, which takes care of diagonal entries, among others), and

$$\int \psi_j(x)\,\epsilon\,\psi_k(x)\,d\,x = 0.$$

whenever $|j - k| > 1$, which targets entries outside of the tridiagonal. Define

$$\varphi_n(x) = \frac{1}{c_n}H_n(x)\,\exp(-x^2/2) \quad \text{for} \quad c_n = \sqrt{2^n n! \sqrt{\pi}}$$

where the H_n are the usual Hermite polynomials defined by the orthogonality condition

$$\int H_j(x) \, H_k(x) \, e^{-x^2} \, d\,x = c_j^2 \, \delta_{j,k}.$$

It follows that

$$\int \varphi_j(x) \, \varphi_k(x) \, d\,x = \delta_{j,k}.$$

Now let

$$\psi_{2n+1}(x) = \frac{d}{d\,x} \, \varphi_{2n}(x) \qquad \psi_{2n}(x) = \varphi_{2n}(x). \tag{7.14}$$

This definition satisfies our earlier requirement that $\psi_j = p_j \, w$ for

$$w(x) = \exp\left(-x^2/2\right).$$

In this case for example

$$p_{2n}(x) = \frac{1}{c_n} \, H_{2n}(x).$$

With ϵ defined as in (6.2), and recalling that, if D denote the operator that acts by differentiation with respect to x, then $D\,\epsilon = \epsilon\,D = I$, it follows that

$$\int \psi_{2m}(x)\epsilon\,\psi_{2n+1}(x)d\,x = \int \varphi_{2m}(x)\,\epsilon\,\frac{d}{d\,x}\,\varphi_{2n+1}(x)\,d\,x$$

$$= \int \varphi_{2m}(x)\,\varphi_{2n+1}(x)\,d\,x$$

$$= \int \varphi_{2m}(x)\,\varphi_{2n+1}(x)\,d\,(x)$$

$$= \delta_{m,n},$$

as desired. Similarly, integration by parts gives

$$\int \psi_{2m+1}(x)\epsilon\,\psi_{2n}(x)d\,x = \int \frac{d}{d\,x}\,\varphi_{2m}(x)\,\epsilon\,\varphi_{2n}(x)\,d\,x$$

$$= -\int \varphi_{2m}(x)\,\varphi_{2n}(x)\,d\,x$$

$$= -\int \varphi_{2m}(x)\,\varphi_{2n+1}(x)\,d\,(x)$$

$$= -\delta_{m,n}.$$

Also ψ_{2n} is even since H_{2n} and φ_{2n} are. Similarly, ψ_{2n+1} is odd. It follows that $\epsilon\psi_{2n}$, and $\epsilon\psi_{2n+1}$, are respectively odd and even functions. From these observations, we obtain

$$\int \psi_{2n}(x)\,\epsilon\,\psi_{2m}(x)d\,x = 0,$$

since the integrand is a product of an odd and an even function. Similarly

$$\int \psi_{2n+1}(x)\,\epsilon\,\psi_{2m+1}(x)d\,x = 0.$$

Finally it is easy to see that if $|j - k| > 1$, then

$$\int \psi_j(x)\,\epsilon\,\psi_k(x)d\,x = 0.$$

Indeed both differentiation and the action of ϵ can only "shift" the indices by 1. Thus by orthogonality of the φ_j, this integral will always be 0. Thus by our choice in (7.14), we force the matrix

$$M = \left(\int \psi_j(x)\,\epsilon\,\psi_k(x)\,d\,x\right)_{0\le j,k\le N-1}$$

to be the direct sum of $\frac{N}{2}$ copies of

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This means $M^{-1} = -M$ where $M^{-1} = \{\mu_{j,k}\}$. Moreover, $\mu_{j,k} = 0$ if j, k have the same parity or $|j - k| > 1$, and $\mu_{2j,2k+1} = \delta_{jk} = -\mu_{2k+1,2j}$ for $j, k = 0.\ldots, \frac{N}{2} - 1$. Therefore

$$S_N(x,y) = -\sum_{j,k=0}^{N-1} \psi_j(x)\mu_{jk}\epsilon\,\psi_k(y)$$

$$= -\sum_{j=0}^{N/2-1} \psi_{2j}(x)\,\epsilon\,\psi_{2j+1}(y) + \sum_{j=0}^{N/2-1} \psi_{2j+1}(x)\,\epsilon\,\psi_{2j}(y)$$

$$= \left[\sum_{j=0}^{N/2-1} \varphi_{2j}\left(\frac{x}{\sigma}\right)\varphi_{2j}\left(\frac{y}{\sigma}\right) - \sum_{j=0}^{N/2-1} \frac{d}{d\,x}\varphi_{2j}\left(\frac{x}{\sigma}\right)\epsilon\,\varphi_{2j}\left(\frac{y}{\sigma}\right)\right].$$

Manipulations similar to those in the $\beta = 4$ case (see (6.8) through (6.13)) yield

$$S_N(x, y) = \left[\sum_{j=0}^{N-1} \varphi_j(x) \varphi_j(y) - \sqrt{\frac{N}{2}} \varphi_{N-1}(x) (\epsilon \varphi_N)(y) \right].$$

We redefine

$$S_N(x, y) = \sum_{j=0}^{N-1} \varphi_j(x) \varphi_j(y) = S_N(y, x),$$

so that the top left entry of $K_{1,N}(x, y)$ is

$$S_N(x, y) + \sqrt{\frac{N}{2}} \varphi_{N-1}(x) (\epsilon \varphi_N)(y).$$

If S_N is the operator with kernel $S_N(x, y)$ then integration by parts gives

$$S_N D f = \int S(x, y) \frac{d}{dy} f(y) \, dy = \int \left(-\frac{d}{dy} S_N(x, y) \right) f(y) \, dy,$$

so that $-\frac{d}{dy} S_N(x, y)$ is in fact the kernel of $S_N D$. Therefore (7.12) now holds with $K_{1,N}$ being the integral operator with matrix kernel $K_{1,N}(x, y)$ whose (i, j)–entry $K_{1,N}^{(i,j)}(x, y)$ is given by

$$K_{1,N}^{(1,1)}(x, y) = \left[S_N(x, y) + \sqrt{\frac{N}{2}} \varphi_{N-1}(x) (\epsilon \varphi_N)(y) \right],$$

$$K_{1,N}^{(1,2)}(x, y) = \left[S D_N(x, y) - \frac{d}{dy} \left(\sqrt{\frac{N}{2}} \varphi_{N-1}(x) (\epsilon \varphi_N)(y) \right) \right],$$

$$K_{1,N}^{(2,1)}(x, y) = \epsilon \left[S_N(x, y) + \sqrt{\frac{N}{2}} \varphi_{N-1}(x) (\epsilon \varphi_N)(y) - 1 \right],$$

$$K_{1,N}^{(2,2)}(x, y) = \left[S_N(x, y) + \sqrt{\frac{N}{2}} (\epsilon \varphi_N)(x) \varphi_{N-1}(y) \right].$$

Define

$$\varphi(x) = \left(\frac{N}{2} \right)^{1/4} \varphi_N(x), \qquad \psi(x) = \left(\frac{N}{2} \right)^{1/4} \varphi_{N-1}(x),$$

so that

$$K_{1,N}^{(1,1)}(x,y) = \chi(x) \left[S_N(x,y) + \psi(x)\,\epsilon\,\varphi(y) \right] \chi(y),$$
$$K_{1,N}^{(1,2)}(x,y) = \chi(x) \left[SD_N(x,y) - \psi(x)\,\varphi(y) \right] \chi(y),$$
$$K_{1,N}^{(2,1)}(x,y) = \chi(x) \left[\epsilon S_N(x,y) + \epsilon\,\psi(x)\,\epsilon\,\varphi(y) - \epsilon(x-y) \right] \chi(y),$$
$$K_{1,N}^{(2,2)}(x,y) = \chi(x) \left[S_N(x,y) + \epsilon\,\varphi(x)\,\psi(y) \right] \chi(y).$$

Note that

$$\chi\,(S + \psi \otimes \epsilon\,\varphi)\,\chi \doteq K_{1,N}^{(1,1)}(x,y),$$
$$\chi\,(SD - \psi \otimes \varphi)\,\chi \doteq K_{1,N}^{(1,2)}(x,y),$$
$$\chi\,(\epsilon\,S + \epsilon\,\psi \otimes \epsilon\,\varphi - \epsilon)\,\chi \doteq K_{1,N}^{(2,1)}(x,y),$$
$$\chi\,(S + \epsilon\,\varphi \otimes \epsilon\,\psi)\,\chi \doteq K_{1,N}^{(2,2)}(x,y).$$

Hence

$$K_{1,N} = \chi \begin{pmatrix} S + \psi \otimes \epsilon\,\varphi & SD - \psi \otimes \varphi \\ \epsilon\,S + \epsilon\,\psi \otimes \epsilon\,\varphi - \epsilon & S + \epsilon\,\varphi \otimes \epsilon\,\psi \end{pmatrix} \chi.$$

Note that this is identical to the corresponding operator for $\beta = 1$ obtained by Tracy and Widom in [33], the only difference being that φ, ψ, and hence also S, are redefined to depend on λ.

7.3 Edge scaling

7.3.1 Reduction of the determinant

The above determinant is that of an operator on $L^2(J) \oplus L^2(J)$. Our first task will be to rewrite these determinants as those of operators on $L^2(J)$. This part follows exactly the proof in [33]. To begin, note that

$$[S,D] = \varphi \otimes \psi + \psi \otimes \varphi \tag{7.15}$$

so that (using the fact that $D\,\epsilon = \epsilon\,D = I$)

$$
\begin{aligned}
[\epsilon, S] &= \epsilon\,S - S\,\epsilon \\
&= \epsilon\,S\,D\,\epsilon - \epsilon\,D\,S\,\epsilon = \epsilon\,[S,D]\,\epsilon \\
&= \epsilon\,\varphi \otimes \psi\,\epsilon + \epsilon\,\psi \otimes \varphi\,\epsilon \\
&= \epsilon\,\varphi \otimes \epsilon^t\,\psi + \epsilon\,\psi \otimes \epsilon^t\,\varphi \\
&= -\epsilon\,\varphi \otimes \epsilon\,\psi - \epsilon\,\psi \otimes \epsilon\,\varphi, \tag{7.16}
\end{aligned}
$$

where the last equality follows from the fact that $\epsilon^t = -\epsilon$. We thus have

$$
\begin{aligned}
D\left(\epsilon\, S + \epsilon\,\psi \otimes \epsilon\,\varphi\right) &= S + \psi \otimes \epsilon\,\varphi, \\
D\left(\epsilon\, S\, D - \epsilon\,\psi \otimes \varphi\right) &= S\, D - \psi \otimes \varphi.
\end{aligned}
$$

The expressions on the right side are the top entries of $K_{1,N}$. Thus the first row of $K_{1,N}$ is, as a vector,

$$
D\left(\epsilon\, S + \epsilon\,\psi \otimes \epsilon\,\varphi, \epsilon\, S\, D - \epsilon\,\psi \otimes \varphi\right).
$$

Now (7.16) implies that

$$
\epsilon\, S + \epsilon\,\psi \otimes \epsilon\,\varphi = S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi.
$$

Similarly (7.15) gives

$$
\epsilon\,[\, S\,,D\,] = \epsilon\,\varphi \otimes \psi + \epsilon\,\psi \otimes \varphi,
$$

so that

$$
\epsilon\, S\, D - \epsilon\,\psi \otimes \varphi = \epsilon\, D\, S + \epsilon\,\varphi \otimes \psi = S + \epsilon\,\varphi \otimes \psi.
$$

Using these expressions we can rewrite the first row of $K_{1,N}$ as

$$
D\left(S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi, S + \epsilon\,\varphi \otimes \psi\right).
$$

Applying ϵ to this expression shows the second row of $K_{1,N}$ is given by

$$
\left(\epsilon\, S - \epsilon + \epsilon\,\psi \otimes \epsilon\,\varphi, S + \epsilon\,\varphi \otimes \psi\right)
$$

Now use (7.16) to show the second row of $K_{1,N}$ is

$$
\left(S\,\epsilon - \epsilon + \epsilon\,\varphi \otimes \epsilon\,\psi, S + \epsilon\,\varphi \otimes \psi\right).
$$

Therefore,

$$
\begin{aligned}
K_{1,N} &= \chi\left(\begin{array}{cc} D\left(S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi\right) & D\left(S + \epsilon\,\varphi \otimes \psi\right) \\ S\,\epsilon - \epsilon + \epsilon\,\varphi \otimes \epsilon\,\psi & S + \epsilon\,\varphi \otimes \psi \end{array}\right)\chi \\
&= \left(\begin{array}{cc} \chi\, D & 0 \\ 0 & \chi \end{array}\right)\left(\begin{array}{cc} \left(S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi\right)\chi & \left(S + \epsilon\,\varphi \otimes \psi\right)\chi \\ \left(S\,\epsilon - \epsilon + \epsilon\,\varphi \otimes \epsilon\,\psi\right)\chi & \left(S + \epsilon\,\varphi \otimes \psi\right)\chi \end{array}\right).
\end{aligned}
$$

Since $K_{1,N}$ is of the form AB, we can use the fact that $\det(I - AB) = \det(I - BA)$ and deduce that $D_{1,N}(s, \lambda)$ is unchanged if instead we take $K_{1,N}$ to be

$$
\begin{aligned}
K_{1,N} &= \begin{pmatrix} (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\chi & (S + \epsilon\,\varphi \otimes \psi)\,\chi \\ (S\,\epsilon - \epsilon + \epsilon\,\varphi \otimes \epsilon\,\psi)\,\chi & (S + \epsilon\,\varphi \otimes \psi)\,\chi \end{pmatrix} \begin{pmatrix} \chi\,D & 0 \\ 0 & \chi \end{pmatrix} \\
&= \begin{pmatrix} (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\chi\,D & (S + \epsilon\,\varphi \otimes \psi)\,\chi \\ (S\,\epsilon - \epsilon + \epsilon\,\varphi \otimes \epsilon\,\psi)\,\chi\,D & (S + \epsilon\,\varphi \otimes \psi)\,\chi \end{pmatrix}.
\end{aligned}
$$

Therefore

$$
D_{1,N}(s, \lambda) = \det \begin{pmatrix} I - (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\lambda\,\chi\,D & -(S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi \\ -(S\,\epsilon - \epsilon + \epsilon\,\varphi \otimes \epsilon\,\psi)\,\lambda\,\chi\,D & I - (S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi \end{pmatrix}.
$$

(7.17)

Now we perform row and column operations on the matrix to simplify it, which do not change the Fredholm determinant. Justification of these operations is given in [33]. We start by subtracting row 1 from row 2 to get

$$
\begin{pmatrix} I - (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\lambda\,\chi\,D & -(S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi \\ -I + \epsilon\,\lambda\,\chi\,D & I \end{pmatrix}.
$$

Next, adding column 2 to column 1 yields

$$
\begin{pmatrix} I - (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\lambda\,\chi\,D - (S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi & -(S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi \\ \epsilon\,\lambda\,\chi\,D & I \end{pmatrix}.
$$

Then right-multiply column 2 by $-\epsilon\,\lambda\,\chi\,D$ and add it to column 1, and multiply row 2 by $S + \epsilon\,\varphi \otimes \psi$ and add it to row 1 to arrive at

$$
\det \begin{pmatrix} I - (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\lambda\,\chi\,D + (S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi\,(\epsilon\,\lambda\,\chi\,D - I) & 0 \\ 0 & I \end{pmatrix}.
$$

Thus the determinant we want equals the determinant of

$$
I - (S\,\epsilon - \epsilon\,\varphi \otimes \epsilon\,\psi)\,\lambda\,\chi\,D + (S + \epsilon\,\varphi \otimes \psi)\,\lambda\,\chi\,(\epsilon\,\lambda\,\chi\,D - I). \qquad (7.18)
$$

So we have reduced the problem from the computation of the Fredholm determinant of an operator on $L^2(J) \times L^2(J)$, to that of an operator on $L^2(J)$.

7.3.2 Differential equations

Next we want to write the operator in (7.18) in the form

$$(I - K_{2,N}) \left(I - \sum_{i=1}^{L} \alpha_i \otimes \beta_i \right), \qquad (7.19)$$

where the α_i and β_i are functions in $L^2(J)$. In other words, we want to rewrite the determinant for the GOE case as a finite dimensional perturbation of the corresponding GUE determinant. The Fredholm determinant of the product is then the product of the determinants. The limiting form for the GUE part is already known, and we can just focus on finding a limiting form for the determinant of the finite dimensional piece. It is here that the proof must be modified from that in [33]. A little simplification of (7.18) yields

$$I - \lambda S \chi - \lambda S (1 - \lambda \chi) \epsilon \chi D - \lambda (\epsilon \varphi \otimes \chi \psi) - \lambda (\epsilon \varphi \otimes \psi)(1 - \lambda \chi) \epsilon \chi D.$$

Writing $\epsilon [\chi, D] + \chi$ for $\epsilon \chi D$ and simplifying $(1 - \lambda \chi) \chi$ to $(1 - \lambda) \chi$ gives

$$
\begin{aligned}
I &- \lambda S \chi - \lambda (1 - \lambda) S \chi - \lambda (\epsilon \varphi \otimes \chi \psi) - \lambda (1 - \lambda)(\epsilon \varphi \otimes \chi \psi) \\
&- \lambda S (1 - \lambda \chi) \epsilon [\chi, D] - \lambda (\epsilon \varphi \otimes \psi)(1 - \lambda \chi) \epsilon [\chi, D] \\
= I &- (2\lambda - \lambda^2) S \chi - (2\lambda - \lambda^2)(\epsilon \varphi \otimes \chi \psi) - \lambda S (1 - \lambda \chi) \epsilon [\chi, D] \\
&- \lambda (\epsilon \varphi \otimes \psi)(1 - \lambda \chi) \epsilon [\chi, D].
\end{aligned}
$$

Define $\tilde{\lambda} = 2\lambda - \lambda^2$ and let $\sqrt{\tilde{\lambda}} \varphi \to \varphi$, and $\sqrt{\tilde{\lambda}} \psi \to \psi$ so that $\tilde{\lambda} S \to S$ and (7.18) goes to

$$
\begin{aligned}
I - S \chi &- (\epsilon \varphi \otimes \chi \psi) - \frac{\lambda}{\tilde{\lambda}} S (1 - \lambda \chi) \epsilon [\chi, D] \\
&- \frac{\lambda}{\tilde{\lambda}} (\epsilon \varphi \otimes \psi)(1 - \lambda \chi) \epsilon [\chi, D].
\end{aligned}
$$

Now we define $R := (I - S \chi)^{-1} S \chi = (I - S \chi)^{-1} - I$ (the resolvent operator of $S \chi$), whose kernel we denote by $R(x, y)$, and $Q_\epsilon := (I - S \chi)^{-1} \epsilon \varphi$. Then (7.18) factors into

$$A = (I - S \chi) B.$$

where B is

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$$I-(Q_\epsilon \otimes \chi\,\psi) - \frac{\lambda}{\tilde{\lambda}}\,(I+R)\,S\,(1-\lambda\,\chi)\,\epsilon\,[\,\chi\,,D\,]$$

$$-\frac{\lambda}{\tilde{\lambda}}\,(Q_\epsilon \otimes \psi)\,(1-\lambda\,\chi)\,\epsilon\,[\,\chi\,,D\,], \qquad \lambda \neq 1.$$

Hence

$$D_{1,N}(s,\lambda) = D_{2,N}(s,\tilde{\lambda})\,\det(B).$$

Note that because of the change of variable $\tilde{\lambda}\,S \to S$, we are in effect factoring $I - (2\lambda - \lambda^2)\,S$, rather that $I - \lambda\,S$ as we did in the $\beta = 4$ case. The fact that we factored $I - (2\lambda - \lambda^2)\,S\,\chi$ as opposed to $I - \lambda\,S\,\chi$ is crucial here for it is what makes B finite rank. If we had factored $I - \lambda\,S\,\chi$ instead, B would have been

$$B = I - \lambda \sum_{k=1}^{2}(-1)^k\,(S + R\,S)\,(I - \lambda\,\chi)\,\epsilon_k \otimes \delta_k - \lambda\,(I+R)\,\epsilon\,\varphi \otimes \chi\,\psi$$

$$-\lambda \sum_{k=1}^{2}(-1)^k\,(\,\psi\,,(I - \lambda\,\chi)\,\epsilon_k\,)\,((I+R)\,\epsilon\,\varphi) \otimes \delta_k$$

$$-\lambda\,(1-\lambda)\,(S + R\,S)\,\chi - \lambda\,(1-\lambda)\,((I+R)\,\epsilon\,\varphi) \otimes \chi\,\psi$$

The first term on the last line is not finite rank, and the methods we have used previously in the $\beta = 4$ case would not work here. It is also interesting to note that these complications disappear when we are dealing with the case of the largest eigenvalue; then is no differentiation with respect to λ, and we just set $\lambda = 1$ in all these formulae. All the new troublesome terms vanish! In order to find $\det(B)$ we use the identity

$$\epsilon\,[\,\chi\,,D\,] = \sum_{k=1}^{2m}(-1)^k\,\epsilon_k \otimes \delta_k, \tag{7.20}$$

where ϵ_k and δ_k are the functions $\epsilon(x - a_k)$ and $\delta(x - a_k)$ respectively, and the a_k are the endpoints of the (disjoint) intervals considered, $J = \cup_{k=1}^{m}(a_{2k-1}, a_{2k})$. We also make use of the fact that

$$a \otimes b \cdot c \otimes d = (\,b\,,c\,) \cdot a \otimes d \tag{7.21}$$

where $(\,.\,,.\,)$ is the usual L^2–inner product. Therefore

$$(Q_\epsilon \otimes \psi) (1 - \lambda \chi) \, \epsilon \, [\chi, D] = \sum_{k=1}^{2m} (-1)^k Q_\epsilon \otimes \psi \cdot (1 - \lambda \chi) \, \epsilon_k \otimes \delta_k$$

$$= \sum_{k=1}^{2m} (-1)^k \, (\, \psi \, , (1 - \lambda \chi) \, \epsilon_k \,) \, Q_\epsilon \otimes \delta_k.$$

It follows that

$$\frac{D_{1,N}(s, \lambda)}{D_{2,N}(s, \tilde{\lambda})}$$

equals the determinant of

$$I - Q_\epsilon \otimes \chi \psi - \frac{\lambda}{\tilde{\lambda}} \sum_{k=1}^{2m} (-1)^k \left[(S + R\,S) (1 - \lambda \chi) \, \epsilon_k + (\, \psi \, , (1 - \lambda \chi) \, \epsilon_k \,) \, Q_\epsilon \right] \otimes \delta_k.$$

$$(7.22)$$

We now specialize to the case of one interval $J = (t, \infty)$, so $m = 1$, $a_1 = t$ and $a_2 = \infty$. We write $\epsilon_t = \epsilon_1$, and $\epsilon_\infty = \epsilon_2$, and similarly for δ_k. Writing the terms in the summation and using the facts that

$$\epsilon_\infty = -\frac{1}{2}, \tag{7.23}$$

and

$$(1 - \lambda \chi) \, \epsilon_t = -\frac{1}{2} (1 - \lambda \chi) + (1 - \lambda \chi) \, \chi, \tag{7.24}$$

then yields

$$I - Q_\epsilon \otimes \chi \psi - \frac{\lambda}{2\tilde{\lambda}} \left[(S + R\,S) (1 - \lambda \chi) + (\, \psi \, , (1 - \lambda \chi)) \, Q_\epsilon \right] \otimes (\delta_t - \delta_\infty)$$

$$+ \frac{\lambda}{\tilde{\lambda}} \left[(S + R\,S) (1 - \lambda \chi) \, \chi + (\, \psi \, , (1 - \lambda \chi) \, \chi \,) \, Q_\epsilon \right] \otimes \delta_t$$

which, to simplify notation, we write as

$$I - Q_\epsilon \otimes \chi \psi - \frac{\lambda}{2\tilde{\lambda}} \left[(S + R\,S) (1 - \lambda \chi) + a_{1,\lambda} \, Q_\epsilon \right] \otimes (\delta_t - \delta_\infty)$$

$$+ \frac{\lambda}{\tilde{\lambda}} \left[(S + R\,S) (1 - \lambda \chi) \, \chi + \tilde{a}_{1,\lambda} \, Q_\epsilon \right] \otimes \delta_t,$$

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where
$$a_{1,\lambda} = (\psi, (1 - \lambda\chi)), \qquad \tilde{a}_{1,\lambda} = (\psi, (1 - \lambda\chi)\chi). \qquad (7.25)$$

Now we can use the formula:
$$\det\left(I - \sum_{i=1}^{L} \alpha_i \otimes \beta_i\right) = \det\left(\delta_{jk} - (\alpha_j, \beta_k)\right)_{1 \le j,k \le L} \qquad (7.26)$$

In this case, $L = 3$, and
$$\alpha_1 = Q_\epsilon, \qquad \alpha_2 = \frac{\lambda}{\tilde{\lambda}}\left[(S + RS)(1 - \lambda\chi) + a_{1,\lambda}Q_\epsilon\right],$$

$$\alpha_3 = -\frac{\lambda}{\tilde{\lambda}}\left[(S + RS)(1 - \lambda\chi)\chi + \tilde{a}_{1,\lambda}Q_\epsilon\right],$$

$$\beta_1 = \chi\psi, \qquad \beta_2 = \delta_t - \delta_\infty, \qquad \beta_3 = \delta_t. \qquad (7.27)$$

In order to simplify the notation, define

$$Q(x, \lambda, t) := (I - S\chi)^{-1}\varphi, \qquad P(x, \lambda, t) := (I - S\chi)^{-1}\psi,$$
$$Q_\epsilon(x, \lambda, t) := (I - S\chi)^{-1}\epsilon\varphi, \qquad P_\epsilon(x, \lambda, t) := (I - S\chi)^{-1}\epsilon\psi, \qquad (7.28)$$

$$q_N := Q(t, \lambda, t), \qquad\qquad p_N := P(t, \lambda, t),$$
$$q_\epsilon := Q_\epsilon(t, \lambda, t), \qquad\qquad p_\epsilon := P_\epsilon(t, \lambda, t),$$
$$u_\epsilon := (Q, \chi\epsilon\varphi) = (Q_\epsilon, \chi\varphi), \qquad v_\epsilon := (Q, \chi\epsilon\psi) = (P_\epsilon, \chi\psi),$$
$$\tilde{v}_\epsilon := (P, \chi\epsilon\varphi) = (Q_\epsilon, \chi\varphi), \qquad w_\epsilon := (P, \chi\epsilon\psi) = (P_\epsilon, \chi\psi), \qquad (7.29)$$

$$\mathcal{P}_{1,\lambda} := \int(1 - \lambda\chi)P\,dx, \qquad \tilde{\mathcal{P}}_{1,\lambda} := \int(1 - \lambda\chi)\chi P\,dx,$$

$$\mathcal{Q}_{1,\lambda} := \int(1 - \lambda\chi)Q\,dx, \qquad \tilde{\mathcal{Q}}_{1,\lambda} := \int(1 - \lambda\chi)\chi Q\,dx,$$

$$\mathcal{R}_{1,\lambda} := \int(1 - \lambda\chi)R(x, t)\,dx, \quad \tilde{\mathcal{R}}_{1,\lambda} := \int(1 - \lambda\chi)\chi R(x, t)\,dx. \qquad (7.30)$$

Note that all quantities in (7.29) and (7.30) are functions of t alone. Furthermore, let

$$c_\varphi = \epsilon\varphi(\infty) = \frac{1}{2}\int_{-\infty}^{\infty}\varphi(x)\,dx, \qquad c_\psi = \epsilon\psi(\infty) = \frac{1}{2}\int_{-\infty}^{\infty}\psi(x)\,dx. \qquad (7.31)$$

Recall from the previous section that when $\beta = 1$ we take N to be even. It follows that φ and ψ are even and odd functions respectively. Thus $c_\psi = 0$ for $\beta = 1$, and computation gives

$$c_\varphi = (\pi N)^{1/4} 2^{-3/4 - N/2} \frac{(N!)^{1/2}}{(N/2)!} \sqrt{\lambda}. \tag{7.32}$$

Hence computation yields

$$\lim_{N \to \infty} c_\varphi = \sqrt{\frac{\lambda}{2}}, \tag{7.33}$$

and at $t = \infty$ we have

$$u_\epsilon(\infty) = 0, \quad q_\epsilon(\infty) = c_\varphi$$

$$\mathcal{P}_{1,\lambda}(\infty) = 2\,c_\psi, \quad \mathcal{Q}_{1,\lambda}(\infty) = 2\,c_\varphi, \quad \mathcal{R}_{1,\lambda}(\infty) = 0,$$
$$\tilde{\mathcal{P}}_{1,\lambda}(\infty) = \tilde{\mathcal{Q}}_{1,\lambda}(\infty) = \tilde{\mathcal{R}}_{1,\lambda}(\infty) = 0.$$

Hence

$$(\,\alpha_1\,,\beta_1\,) = \tilde{v}_\epsilon, \quad (\,\alpha_1\,,\beta_2\,) = q_\epsilon - c_\varphi, \quad (\,\alpha_1\,,\beta_3\,) = q_\epsilon, \tag{7.34}$$

$$(\,\alpha_2\,,\beta_1\,) = \frac{\lambda}{2\,\tilde{\lambda}}\,\left[\mathcal{P}_{1,\lambda} - a_{1,\lambda}\,(1 - \tilde{v}_\epsilon)\right], \tag{7.35}$$

$$(\,\alpha_2\,,\beta_2\,) = \frac{\lambda}{2\,\tilde{\lambda}}\,\left[\mathcal{R}_{1,\lambda} + a_{1,\lambda}\,(q_\epsilon - c_\varphi)\right], \tag{7.36}$$

$$(\,\alpha_2\,,\beta_3\,) = \frac{\lambda}{2\,\tilde{\lambda}}\,\left[\mathcal{R}_{1,\lambda} + a_{1,\lambda}\,q_\epsilon\right], \tag{7.37}$$

$$(\,\alpha_3\,,\beta_1\,) = -\frac{\lambda}{\tilde{\lambda}}\,\left[\tilde{\mathcal{P}}_{1,\lambda} - \tilde{a}_{1,\lambda}\,(1 - \tilde{v}_\epsilon)\right], \tag{7.38}$$

$$(\,\alpha_3\,,\beta_2\,) = -\frac{\lambda}{\tilde{\lambda}}\,\left[\tilde{\mathcal{R}}_{1,\lambda} + \tilde{a}_{1,\lambda}\,(q_\epsilon - c_\varphi)\right], \tag{7.39}$$

$$(\,\alpha_3\,,\beta_3\,) = -\frac{\lambda}{\tilde{\lambda}}\,\left[\tilde{\mathcal{R}}_{1,\lambda} + \tilde{a}_{1,\lambda}\,q_\epsilon\right]. \tag{7.40}$$

As an illustration, let us do the computation that led to (7.36) in detail. As in [33], we use the facts that $S^t = S$, and $(S + S\,R^t)\,\mathcal{X} = R$ which can be easily seen by writing $R = \sum_{k=1}^{\infty}(S\,\mathcal{X})^k$. Furthermore we write $R(x, a_k)$ to mean

$$\lim_{\substack{y \to a_k \\ y \in J}} R(x, y).$$

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In general, since all evaluations are done by taking the limits from within J, we can use the identity $\chi \delta_k = \delta_k$ inside the inner products. Thus

$$
\begin{aligned}
(\alpha_2 , \beta_2) &= \frac{\lambda}{\tilde{\lambda}} \left[((S + R\,S) (1 - \lambda\,\chi) , \delta_t - \delta_\infty) + a_{1,\lambda} \, (Q_\epsilon , \delta_t - \delta_\infty) \right] \\
&= \frac{\lambda}{\tilde{\lambda}} \left[((1 - \lambda\,\chi) , (S + R^t\,S) \, (\delta_t - \delta_\infty)) + a_{1,\lambda} \, (Q_\epsilon(t) - Q_\epsilon(\infty)) \right] \\
&= \frac{\lambda}{\tilde{\lambda}} \left[((1 - \lambda\,\chi) , (S + R^t\,S) \, \chi \, (\delta_t - \delta_\infty)) + a_{1,\lambda} \, (q_\epsilon - c_\varphi) \right] \\
&= \frac{\lambda}{\tilde{\lambda}} \left[((1 - \lambda\,\chi) , R(x,t) - R(x,\infty)) + a_{1,\lambda} \, (q_\epsilon - c_\varphi) \right] \\
&= \frac{\lambda}{\tilde{\lambda}} \left[\mathcal{R}_{1,\lambda}(t) - \mathcal{R}_{1,\lambda}(\infty) + a_{1,\lambda} \, (q_\epsilon - c_\varphi) \right] \\
&= \frac{\lambda}{\tilde{\lambda}} \left[\mathcal{R}_{1,\lambda}(t) + a_{1,\lambda} \, (q_\epsilon - c_\varphi) \right] .
\end{aligned}
$$

We want the limit of the determinant

$$
\det \left(\delta_{jk} - (\alpha_j , \beta_k) \right)_{1 \le j,k \le 3} , \tag{7.41}
$$

as $N \to \infty$. In order to get our hands on the limits of the individual terms involved in the determinant, we will find differential equations for them first as in [33]. Row operation on the matrix show that $a_{1,\lambda}$ and $\tilde{a}_{1,\lambda}$ fall out of the determinant; to see this add $\lambda\,a_{1,\lambda}/(2\,\tilde{\lambda})$ times row 1 to row 2 and $\lambda\,\tilde{a}_{1,\lambda}/\tilde{\lambda}$ times row 1 to row 3. So we will not need to find differential equations for them. Our determinant is

$$
\det \begin{pmatrix}
1 - \tilde{v}_\epsilon & -(q_\epsilon - c_\varphi) & -q_\epsilon \\
-\dfrac{\lambda\,\mathcal{P}_{1,\lambda}}{2\,\tilde{\lambda}} & 1 - \dfrac{\lambda\,\mathcal{R}_{1,\lambda}}{2\,\tilde{\lambda}} & -\dfrac{\lambda\,\mathcal{R}_{1,\lambda}}{2\,\tilde{\lambda}} \\
\dfrac{\lambda\,\tilde{\mathcal{P}}_{1,\lambda}}{\tilde{\lambda}} & \dfrac{\lambda\,\tilde{\mathcal{R}}_{1,\lambda}}{\tilde{\lambda}} & 1 + \dfrac{\lambda\,\tilde{\mathcal{R}}_{1,\lambda}}{\tilde{\lambda}}
\end{pmatrix} . \tag{7.42}
$$

Proceeding as in [33] we find the following differential equations

$$\frac{d}{dt} u_\epsilon = q_N \, q_\epsilon, \qquad\qquad \frac{d}{dt} q_\epsilon = q_N - q_N \, \tilde{v}_\epsilon - p_N \, u_\epsilon, \qquad (7.43)$$

$$\frac{d}{dt} \mathcal{Q}_{1,\lambda} = q_N \left(\lambda - \mathcal{R}_{1,\lambda}\right), \qquad\qquad \frac{d}{dt} \mathcal{P}_{1,\lambda} = p_N \left(\lambda - \mathcal{R}_{1,\lambda}\right), \qquad (7.44)$$

$$\frac{d}{dt} \mathcal{R}_{1,\lambda} = -p_N \, \mathcal{Q}_{1,\lambda} - q_N \, \mathcal{P}_{1,\lambda}, \qquad \frac{d}{dt} \tilde{\mathcal{R}}_{1,\lambda} = -p_N \, \tilde{\mathcal{Q}}_{1,\lambda} - q_N \, \tilde{\mathcal{P}}_{1,\lambda}, \qquad (7.45)$$

$$\frac{d}{dt} \tilde{\mathcal{Q}}_{1,\lambda} = q_N \left(\lambda - 1 - \tilde{\mathcal{R}}_{1,\lambda}\right), \qquad \frac{d}{dt} \tilde{\mathcal{P}}_{1,\lambda} = p_N \left(\lambda - 1 - \tilde{\mathcal{R}}_{1,\lambda}\right). \qquad (7.46)$$

Let us derive the first equation in (7.44) for example. From [31] (equation 2.17), we have

$$\frac{\partial Q}{\partial t} = -R(x,t) \, q_N.$$

Therefore

$$\frac{\partial \mathcal{Q}_{1,\lambda}}{\partial t} = \frac{d}{dt} \left[\int_{-\infty}^t Q(x,t) \, dx - (1-\lambda) \int_\infty^t Q(x,t) \, dx \right]$$

$$= q_N + \int_{-\infty}^t \frac{\partial Q}{\partial t} \, dx - (1-\lambda) \left[q_N + \int_\infty^t \frac{\partial Q}{\partial t} \, dx \right]$$

$$= q_N - q_N \int_{-\infty}^t R(x,t) \, dx - (1-\lambda) \, q_N + (1-\lambda) \, q_N \int_\infty^t R(x,t) \, dx$$

$$= \lambda \, q_N - q_N \int_{-\infty}^\infty (1-\lambda) \, R(x,t) \, dx$$

$$= \lambda \, q_N - q_N \, \mathcal{R}_{1,\lambda} = q_N \left(\lambda - \mathcal{R}_{1,\lambda}\right).$$

Now we change variable from t to s where $t = \tau(s) = 2 \, \sigma \, \sqrt{N} + \frac{\sigma \, s}{N^{1/6}}$. Then we take the limit $N \to \infty$, denoting the limits of $q_\epsilon, \mathcal{P}_{1,\lambda}, \mathcal{Q}_{1,\lambda}, \mathcal{R}_{1,\lambda}, \tilde{\mathcal{P}}_{1,\lambda}, \tilde{\mathcal{Q}}_{1,\lambda}, \tilde{\mathcal{R}}_{1,\lambda}$ and the common limit of u_ϵ and $\tilde{v}_\epsilon$ respectively by $\overline{q}, \overline{\mathcal{P}}_{1,\lambda}, \overline{\mathcal{Q}}_{1,\lambda}, \overline{\mathcal{R}}_{1,\lambda}, \overline{\overline{\mathcal{P}}}_{1,\lambda}, \overline{\overline{\mathcal{Q}}}_{1,\lambda}, \overline{\overline{\mathcal{R}}}_{1,\lambda}$ and $\overline{u}$. We eliminate $\overline{\mathcal{Q}}_{1,\lambda}$ and $\overline{\overline{\mathcal{Q}}}_{1,\lambda}$ by using the facts that $\overline{\mathcal{Q}}_{1,\lambda} = \overline{\mathcal{P}}_{1,\lambda} + \lambda \sqrt{2}$ and $\overline{\overline{\mathcal{Q}}}_{1,\lambda} = \overline{\overline{\mathcal{P}}}_{1,\lambda}$. These limits hold uniformly for bounded s so we can interchange $\lim$ and $\frac{d}{ds}$. Also $\lim_{N \to \infty} N^{-1/6} q_N = \lim_{N \to \infty} N^{-1/6} p_N = q$, where q is as in (3.10). We obtain the systems

$$\frac{d}{ds} \overline{u} = -\frac{1}{\sqrt{2}} q \overline{q}, \qquad \frac{d}{ds} \overline{q} = \frac{1}{\sqrt{2}} q \left(1 - 2 \overline{u}\right), \qquad (7.47)$$

$$\frac{d}{ds}\overline{\mathcal{P}}_{1,\lambda} = -\frac{1}{\sqrt{2}}\,q\left(\overline{\mathcal{R}}_{1,\lambda} - \lambda\right), \qquad \frac{d}{ds}\overline{\mathcal{R}}_{1,\lambda} = -\frac{1}{\sqrt{2}}\,q\left(2\,\overline{\mathcal{P}}_{1,\lambda} + \sqrt{2\tilde{\lambda}}\right),$$

$$\text{(7.48)}$$

$$\frac{d}{ds}\overline{\overline{\mathcal{P}}}_{1,\lambda} = \frac{1}{\sqrt{2}}\,q\left(1 - \lambda - \overline{\overline{\mathcal{R}}}_{1,\lambda}\right), \qquad \frac{d}{ds}\overline{\overline{\mathcal{R}}}_{1,\lambda} = -q\sqrt{2}\,\overline{\overline{\mathcal{P}}}_{1,\lambda}. \qquad \text{(7.49)}$$

The change of variables $s \to \mu = \int_s^\infty q(x,\lambda)\,dx$ transforms these systems into constant coefficient ordinary differential equations

$$\frac{d}{d\mu}\overline{u} = \frac{1}{\sqrt{2}}\,\overline{q}, \qquad \frac{d}{d\mu}\overline{q} = -\frac{1}{\sqrt{2}}\left(1 - 2\,\overline{u}\right), \qquad \text{(7.50)}$$

$$\frac{d}{d\mu}\overline{\mathcal{P}}_{1,\lambda} = \frac{1}{\sqrt{2}}\left(\overline{\mathcal{R}}_{1,\lambda} - \lambda\right), \qquad \frac{d}{d\mu}\overline{\mathcal{R}}_{1,\lambda} = \frac{1}{\sqrt{2}}\left(2\,\overline{\mathcal{P}}_{1,\lambda} + \sqrt{2\tilde{\lambda}}\right), \quad \text{(7.51)}$$

$$\frac{d}{d\mu}\overline{\overline{\mathcal{P}}}_{1,\lambda} = -\frac{1}{\sqrt{2}}\left(1 - \lambda - \overline{\overline{\mathcal{R}}}_{1,\lambda}\right), \qquad \frac{d}{d\mu}\overline{\overline{\mathcal{R}}}_{1,\lambda} = \sqrt{2}\,\overline{\overline{\mathcal{P}}}_{1,\lambda}. \qquad \text{(7.52)}$$

Since $\lim_{s\to\infty}\mu = 0$, corresponding to the boundary values at $t = \infty$ which we found earlier for $\mathcal{P}_{1,\lambda}, \mathcal{R}_{1,\lambda}, \tilde{\mathcal{P}}_{1,\lambda}, \tilde{\mathcal{R}}_{1,\lambda}$, we now have initial values at $\mu = 0$. Therefore

$$\overline{\mathcal{P}}_{1,\lambda}(0) = \overline{\mathcal{R}}_{1,\lambda}(0) = \overline{\overline{\mathcal{P}}}_{1,\lambda}(0) = \overline{\overline{\mathcal{R}}}_{1,\lambda}(0) = 0. \qquad \text{(7.53)}$$

We use this to solve the systems and get

$$\overline{q} = \frac{\sqrt{\tilde{\lambda}} - 1}{2\sqrt{2}}\,e^\mu + \frac{\sqrt{\tilde{\lambda}} + 1}{2\sqrt{2}}\,e^{-\mu}, \qquad \text{(7.54)}$$

$$\overline{u} = \frac{\sqrt{\tilde{\lambda}} - 1}{4}\,e^\mu - \frac{\sqrt{\tilde{\lambda}} + 1}{4}\,e^{-\mu} + \frac{1}{2}, \qquad \text{(7.55)}$$

$$\overline{\mathcal{P}}_{1,\lambda} = \frac{\sqrt{\tilde{\lambda}} - \lambda}{2\sqrt{2}}\,e^\mu + \frac{\sqrt{\tilde{\lambda}} + \lambda}{2\sqrt{2}}\,e^{-\mu} - \sqrt{\frac{\tilde{\lambda}}{2}}, \qquad \text{(7.56)}$$

$$\overline{\mathcal{R}}_{1,\lambda} = \frac{\sqrt{\tilde{\lambda}} - \lambda}{2}\,e^\mu - \frac{\sqrt{\tilde{\lambda}} + \lambda}{2}\,e^{-\mu} + \lambda, \qquad \text{(7.57)}$$

$$\overline{\overline{\mathcal{P}}}_{1,\lambda} = \frac{1-\lambda}{2\sqrt{2}}\left(e^\mu - e^{-\mu}\right), \qquad \overline{\overline{\mathcal{R}}}_{1,\lambda} = \frac{1-\lambda}{2}\left(e^\mu + e^{-\mu} - 2\right). \qquad \text{(7.58)}$$

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Substituting these expressions into the determinant gives (3.12), namely

$$D_1(s, \lambda) = D_2(s, \tilde{\lambda}) \frac{\lambda - 1 - \cosh \mu(s, \tilde{\lambda}) + \sqrt{\tilde{\lambda}} \sinh \mu(s, \tilde{\lambda})}{\lambda - 2}, \qquad (7.59)$$

where $D_\beta = \lim_{N \to \infty} D_{\beta,N}$. As mentioned in Section 2.1, the functional form of the $\beta = 1$ limiting determinant is very different from what one would expect, unlike in the $\beta = 4$ case. Also noteworthy is the dependence on $\tilde{\lambda} = 2\lambda - \lambda^2$ instead of just λ. However one should also note that when λ is set equal to 1, then $\tilde{\lambda} = \lambda = 1$. Hence in the largest eigenvalue case, where there is no prior differentiation with respect to λ, and λ is just set to 1, a great deal of simplification occurs. The above formula then nicely reduces to the $\beta = 1$ Tracy-Widom distribution.

8 An Interlacing Property

The following series of lemmas establish Corollary (3.15):

Lemma 8.1. *Define*

$$a_j = \frac{d^j}{d\lambda^j} \sqrt{\frac{\lambda}{2 - \lambda}} \bigg|_{\lambda=1}. \qquad (8.1)$$

Then a_j satisfies the following recursion

$$a_j = \begin{cases} 1 & \text{if } j = 0, \\ (j - 1)\, a_{j-1} & \text{for } j \geq 1,\ j \text{ even,} \\ j\, a_{j-1} & \text{for } j \geq 1,\ j \text{ odd.} \end{cases} \qquad (8.2)$$

Proof. Consider the expansion of the generating function $f(\lambda) = \sqrt{\frac{\lambda}{2-\lambda}}$ around $\lambda = 1$

$$f(\lambda) = \sum_{j \geq 0} \frac{a_j}{j!} (\lambda - 1)^j = \sum_{j \geq 0} b_j (\lambda - 1)^j.$$

Since $a_j = j!\, b_j$, the statement of the lemma reduces to proving the following recurrence for the b_j

$$b_j = \begin{cases} 1 & \text{if } j = 0, \\ \frac{j-1}{j} b_{j-1} & \text{for } j \geq 1,\ j \text{ even,} \\ b_{j-1} & \text{for } j \geq 1,\ j \text{ odd.} \end{cases} \qquad (8.3)$$

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Let

$$f^{even}(\lambda) = \frac{1}{2}\left(\sqrt{\frac{\lambda}{2-\lambda}} + \sqrt{\frac{2-\lambda}{\lambda}}\right), \qquad f^{odd}(\lambda) = \frac{1}{2}\left(\sqrt{\frac{\lambda}{2-\lambda}} - \sqrt{\frac{2-\lambda}{\lambda}}\right).$$

These are the even and odd parts of f relative to the reflection $\lambda - 1 \to -(\lambda - 1)$ or $\lambda \to 2 - \lambda$. Recurrence (8.3) is equivalent to

$$\frac{d}{d\lambda} f^{even}(\lambda) = (\lambda - 1)\frac{d}{d\lambda} f^{odd}(\lambda),$$

which is easily shown to be true. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 8.2. *Define*

$$f(s, \lambda) = 1 - \sqrt{\frac{\lambda}{2-\lambda}} \; \tanh\frac{\mu(s, \tilde{\lambda})}{2}, \qquad\qquad (8.4)$$

for $\tilde{\lambda} = 2\lambda - \lambda^2$. Then

$$\frac{\partial^{2n}}{\partial\lambda^{2n}} f(s, \lambda)\Big|_{\lambda=1} - \frac{1}{2n+1}\frac{\partial^{2n+1}}{\partial\lambda^{2n+1}} f(s, \lambda)\Big|_{\lambda=1} = \left\{ \begin{array}{ll} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{array} \right. \qquad (8.5)$$

Proof. The case $n = 0$ is readily checked. The main ingredient for the general case is Faá di Bruno's formula

$$\frac{d^n}{dt^n} g(h(t)) = \sum \frac{n!}{k_1! \cdots k_n!} \left(\frac{d^k g}{dh^k}(h(t))\right)\left(\frac{1}{1!}\frac{dh}{dt}\right)^{k_1} \cdots \left(\frac{1}{n!}\frac{d^n h}{dt^n}\right)^{k_n}, \quad (8.6)$$

where $k = \sum_{i=1}^{n} k_i$ and the above sum is over all partitions of n, that is all values of $k_1, \ldots, k_n$ such that $\sum_{i=1}^{n} i\,k_i = n$. We apply Faá di Bruno's formula to derivatives of the function $\tanh\frac{\mu(s,\tilde{\lambda})}{2}$, which we treat as some function $g(\tilde{\lambda}(\lambda))$. Notice that for $j \geq 1$, $\frac{d^j\tilde{\lambda}}{d\lambda^j}\big|_{\lambda=1}$ is nonzero only when $j = 2$, in which case it equals -2. Hence, in (8.6), the only term that survives is the one corresponding to the partition all of whose parts equal 2. Thus we have

$$\frac{\partial^{2n-k}}{\partial\lambda^{2n-k}}\tanh\frac{\mu(s,\tilde{\lambda})}{2}\bigg|_{\lambda=1}$$

$$= \left\{ \begin{array}{ll} 0 & \text{if } k = 2j+1, \; j \geq 0 \\[2mm] \frac{(-1)^{n-j}(2n-k)!}{(n-j)!}\frac{\partial^{n-j}}{\partial\tilde{\lambda}^{n-j}}\tanh\frac{\mu(s,\tilde{\lambda})}{2}\bigg|_{\tilde{\lambda}=1} & \text{for } k = 2j, \; j \geq 0 \end{array} \right.$$

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$$
\frac{\partial^{2n-k+1}}{\partial \lambda^{2n+1-k}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \bigg|_{\lambda=1}
$$

$$
= \begin{cases} 0 & \text{if } k = 2j,\ j \geq 0 \\[2mm] \frac{(-1)^{n-j}\,(2\,n+1-k)!}{(n-j)!} \frac{\partial^{n-j}}{\partial \tilde{\lambda}^{n-j}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \bigg|_{\tilde{\lambda}=1} & \text{for } k = 2j+1,\ j \geq 0 \end{cases}
$$

Therefore, recalling the definition of a_j in (8.1) and setting $k = 2\,j$, we obtain

$$
\begin{aligned}
\frac{\partial^{2n}}{\partial \lambda^{2n}} f(s,\lambda) \bigg|_{\lambda=1} &= \sum_{k=0}^{2n} \binom{2\,n}{k} \frac{\partial^k}{\partial \lambda^k} \sqrt{\frac{\lambda}{2-\lambda}} \frac{\partial^{2\,n-k}}{\partial \lambda^{2\,n-k}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \bigg|_{\lambda=1} \\
&= \sum_{j=0}^{n} \frac{(2\,n)!\,(-1)^{n-j}}{(2\,j)!\,(n-j)!} a_{2j} \frac{\partial^{n-j}}{\partial \tilde{\lambda}^{n-j}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \bigg|_{\tilde{\lambda}=1}.
\end{aligned}
$$

Similarly, using $k = 2\,j + 1$ instead yields

$$
\begin{aligned}
\frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} f(s,\lambda) \bigg|_{\lambda=1} &= \sum_{k=0}^{2n+1} \binom{2\,n+1}{k} \frac{\partial^k}{\partial \lambda^k} \sqrt{\frac{\lambda}{2-\lambda}} \frac{\partial^{2\,n+1-k}}{\partial \lambda^{2\,n+1-k}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \bigg|_{\lambda=1} \\
&= (2\,n+1) \sum_{j=0}^{n} \frac{(2\,n)!\,(-1)^{n-j}}{(2\,j)!\,(n-j)!} \frac{a_{2j+1}}{2\,j+1} \frac{\partial^{n-j}}{\partial \tilde{\lambda}^{n-j}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \bigg|_{\tilde{\lambda}=1} \\
&= (2\,n+1) \frac{\partial^{2n}}{\partial \lambda^{2n}} f(s,\lambda) \bigg|_{\lambda=1},
\end{aligned}
$$

since $a_{2j+1}/(2\,j+1) = a_{2j}$. Rearranging this last equality leads to (8.5). $\qquad\square$

Lemma 8.3. *Let $D_1(s,\lambda)$ and $D_4(s,\tilde{\lambda})$ be as in (3.12) and (3.13). Then*

$$
D_1(s,\lambda) = D_4(s,\tilde{\lambda}) \left(1 - \sqrt{\frac{\lambda}{2-\lambda}} \tanh \frac{\mu(s,\tilde{\lambda})}{2} \right)^2. \tag{8.7}
$$

Proof. Using the facts that $-1 - \cosh x = -2 \cosh^2 \frac{x}{2}$, $1 = \cosh^2 x - \sinh^2 x$ and $\sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2}$ we get

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$$
\begin{aligned}
D_1(s,\lambda) \;&=\; \frac{-1-\cosh\mu(s,\tilde\lambda)}{\lambda-2}\,D_2(s,\tilde\lambda) + D_2(s,\tilde\lambda)\,\frac{\lambda+\sqrt{\tilde\lambda}\,\sinh\mu(s,\tilde\lambda)}{\lambda-2}\\[2mm]
&=\; \frac{-2}{\lambda-2}\,D_4(s,\tilde\lambda)\\[2mm]
&\qquad + D_2(s,\tilde\lambda)\,\frac{\lambda\left(\cosh^2\frac{\mu(s,\tilde\lambda)}{2}-\sinh^2\frac{\mu(s,\tilde\lambda)}{2}\right)+\sqrt{\tilde\lambda}\,\sinh\mu(s,\tilde\lambda)}{\lambda-2}\\[2mm]
&=\; D_4(s,\tilde\lambda) + \frac{D_4(s,\tilde\lambda)}{\cosh^2\left(\frac{\mu(s,\lambda)}{2}\right)}\,\frac{\lambda\,\sinh^2\frac{\mu(s,\tilde\lambda)}{2}-\sqrt{\tilde\lambda}\,\sinh\mu(s,\tilde\lambda)}{2-\lambda}\\[2mm]
&=\; D_4(s,\tilde\lambda)\left(1+\frac{\lambda\,\sinh^2\frac{\mu(s,\tilde\lambda)}{2}-2\sqrt{\tilde\lambda}\,\sinh\left(\frac{\mu(s,\lambda)}{2}\right)\cosh\left(\frac{\mu(s,\lambda)}{2}\right)}{(2-\lambda)\cosh^2\left(\frac{\mu(s,\lambda)}{2}\right)}\right)\\[2mm]
&=\; D_4(s,\tilde\lambda)\left(1-2\sqrt{\frac{\lambda}{2-\lambda}}\,\tanh\frac{\mu(s,\tilde\lambda)}{2}+\frac{\lambda}{2-\lambda}\,\tanh^2\frac{\mu(s,\tilde\lambda)}{2}\right)\\[2mm]
&=\; D_4(s,\tilde\lambda)\left(1-\sqrt{\frac{\lambda}{2-\lambda}}\,\tanh\frac{\mu(s,\tilde\lambda)}{2}\right)^2.
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

For notational convenience, define $d_1(s,\lambda)=D_1^{1/2}(s,\lambda)$, and $d_4(s,\lambda)=D_4^{1/2}(s,\lambda)$. Then

Lemma 8.4. *For $n\ge 0$,*

$$
\left[-\frac{1}{(2n+1)!}\frac{\partial^{2n+1}}{\partial\lambda^{2n+1}}+\frac{1}{(2n)!}\frac{\partial^{2n}}{\partial\lambda^{2n}}\right]d_1(s,\lambda)\,\Bigg|_{\lambda=1} = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda^n}d_4(s,\lambda)\,\Bigg|_{\lambda=1}.
$$

Proof. Let

$$
f(s,\lambda)=1-\sqrt{\frac{\lambda}{2-\lambda}}\,\tanh\frac{\mu(s,\tilde\lambda)}{2}
$$

by the previous lemma, we need to show that

$$
\left[-\frac{1}{(2n+1)!}\frac{\partial^{2n+1}}{\partial\lambda^{2n+1}}+\frac{1}{(2n)!}\frac{\partial^{2n}}{\partial\lambda^{2n}}\right]d_4(s,\tilde\lambda)\,f(s,\lambda)\,\Bigg|_{\lambda=1} \qquad (8.8)
$$

equals

$$\frac{(-1)^n}{n!} \frac{\partial^n}{\partial \tilde{\lambda}^n} d_4(s, \tilde{\lambda}) \bigg|_{\lambda=1}.$$

Now formula (8.6) applied to $d_4(s, \tilde{\lambda})$ gives

$$\frac{\partial^k}{\partial \lambda^k} d_4(s, \tilde{\lambda}) \bigg|_{\lambda=1} = \begin{cases} 0 & \text{if } k = 2j+1, \ j \geq 0, \\ \frac{(-1)^j k!}{j!} \frac{\partial^j}{\partial \tilde{\lambda}^j} d_4(s, \tilde{\lambda}) & \text{if } k = 2j, \ j \geq 0. \end{cases}$$

Therefore

$$-\frac{1}{(2n+1)!} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} d_4(s, \tilde{\lambda}) f(s, \lambda) \bigg|_{\lambda=1}$$

$$= -\frac{1}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\partial^k}{\partial \lambda^k} d_4 \frac{\partial^{2n+1-k}}{\partial \lambda^{2n+1-k}} f \bigg|_{\lambda=1}$$

$$= -\sum_{j=0}^{n} \frac{(-1)^j}{(2n-2j+1)! \, j!} \frac{\partial^j}{\partial \tilde{\lambda}^j} d_4 \frac{\partial^{2n-2j+1}}{\partial \lambda^{2n-2j+1}} f \bigg|_{\lambda=1}.$$

Similarly,

$$\frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \lambda^{2n}} d_4(s, \tilde{\lambda}) f(s, \lambda) \bigg|_{\lambda=1} = \frac{1}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} \frac{\partial^k}{\partial \lambda^k} d_4 \frac{\partial^{2n-k}}{\partial \lambda^{2n-k}} f \bigg|_{\lambda=1}$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{(2n-2j)! \, j!} \frac{\partial^j}{\partial \tilde{\lambda}^j} d_4 \frac{\partial^{2n-2j}}{\partial \lambda^{2n-2j}} f \bigg|_{\lambda=1}.$$

Therefore the expression in (8.8) equals

$$\sum_{j=0}^{n} \frac{(-1)^j}{(2n-2j)! \, j!} \frac{\partial^j}{\partial \tilde{\lambda}^j} d_4(s, \tilde{\lambda}) \left[\frac{\partial^{2n-2j}}{\partial \lambda^{2n-2j}} f - \frac{1}{2n-2j+1} \frac{\partial^{2n-2j+1}}{\partial \lambda^{2n-2j+1}} f \right] \bigg|_{\lambda=1}.$$

Now Lemma 8.2 shows that the square bracket inside the summation is zero unless $j = n$, in which case it is 1. The result follows. $\square$

In an inductive proof of Corollary 3.15, the base case $F_4(s, 2) = F_1(s, 1)$ is easily checked by direct calculation. Lemma 8.4 establishes the inductive step in the proof since, with the assumption $F_4(s, n) = F_1(s, 2n)$, it is equivalent to the statement

$$F_4(s, n+1) = F_1(s, 2n+2).$$

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9 Numerics

9.1 Partial derivatives of $q(x, \lambda)$

Let

$$q_n(x) = \left. \frac{\partial^n}{\partial \lambda^n} q(x, \lambda) \right|_{\lambda=1}, \tag{9.1}$$

so that q_0 equals q from (3.8). In order to compute $F_\beta(s, m)$ it is crucial to know q_n with $0 \le n \le m$ accurately. Asymptotic expansions for q_n at $-\infty$ are given in [31]. In particular, we know that, as $t \to +\infty$, $q_0(-t/2)$ is given by

$$\frac{1}{2} \sqrt{t} \left(1 - \frac{1}{t^3} - \frac{73}{2t^6} - \frac{10657}{2t^9} - \frac{13912277}{8t^{12}} + \mathrm{O}\left(\frac{1}{t^{15}} \right) \right),$$

whereas $q_1(-t/2)$ can be expanded as

$$\frac{\exp\left(\frac{1}{3} t^{3/2} \right)}{2\sqrt{2\pi}\, t^{1/4}} \left(1 + \frac{17}{24 t^{3/2}} + \frac{1513}{2^7 3^2 t^3} + \frac{850193}{2^{10} 3^4 t^{9/2}} - \frac{407117521}{2^{15} 3^5 t^6} + \mathrm{O}\left(\frac{1}{t^{15/2}} \right) \right).$$

These expansions are used in the algorithms below.

9.2 Algorithms

Quantities needed to compute $F_\beta(s, m)$, $m = 1, 2$, are not only q_0 and q_1 but also integrals involving q_0, such as

$$I_0 = \int_s^\infty (x - s)\, q_0^2(x)\, d\,x, \quad J_0 = \int_s^\infty q_0(x)\, d\,x. \tag{9.2}$$

Instead of computing these integrals afterward, it is better to include them as variables in a system together with q_0, as suggested in [28]. Therefore all quantities needed are computed in one step, greatly reducing errors, and taking full advantage of the powerful numerical tools in MATLAB$^{\mathrm{TM}}$. Since

$$I_0' = -\int_s^\infty q_0^2(x)\, d\,x, \quad I_0'' = q_0^2, \quad J_0' = -q_0, \tag{9.3}$$

the system closes, and can be concisely written

$$\frac{d}{ds} \begin{pmatrix} q_0 \\ q_0' \\ I_0 \\ I_0' \\ J_0 \end{pmatrix} = \begin{pmatrix} q_0' \\ s\, q_0 + 2q_0^3 \\ I_0' \\ q_0^2 \\ -q_0 \end{pmatrix}. \tag{9.4}$$

We first use the MATLAB™ built–in Runge–Kutta–based ODE solver `ode45` to obtain a first approximation to the solution of (9.4) between $x = 6$, and $x = -8$, with an initial values obtained using the Airy function on the right hand side. Note that it is not possible to extend the range to the left due to the high instability of the solution a little after -8. (This is where the transition region between the three different regimes in the so–called "connection problem" lies. We circumvent this limitation by patching up our solution with the asymptotic expansion to the left of $x = -8$.) The approximation obtained is then used as a trial solution in the MATLAB™ boundary value problem solver `bvp4c`, resulting in an accurate solution vector between $x = 6$ and $x = -10$. Similarly, if we define

$$I_1 = \int_s^\infty (x - s)\, q_0(x)\, q_1(x)\, d\,x, \quad J_1 = \int_s^\infty q_0(x)\, q_1(x)\, d\,x, \qquad (9.5)$$

then we have the first–order system

$$\frac{d}{ds} \begin{pmatrix} q_1 \\ q_1' \\ I_1 \\ I_1' \\ J_1 \end{pmatrix} = \begin{pmatrix} q_1' \\ s\, q_1 + 6q_0^2\, q_1 \\ I_1' \\ q_0\, q_1 \\ -q_0\, q_1 \end{pmatrix}, \qquad (9.6)$$

which can be implemented using `bvp4c` together with a "seed" solution obtained in the same way as for q_0.

The MATLAB™ code is freely available, and may be obtained by contacting the first author.

9.3   Tables

Eigenvalue	Statistic			
	μ	σ	γ_1	γ_2
λ_1	-1.771087	0.901773	0.224084	0.093448
λ_2	-3.675440	0.735214	0.125000	0.021650

Table 1: Mean, standard deviation, skewness and kurtosis data for first two edge–scaled eigenvalues in the $\beta = 2$ Gaussian ensemble. Compare to Table 1 in [35].

Eigenvalue	Statistic			
	μ	σ	γ_1	γ_2
λ_1	-1.206548	1.267941	0.293115	0.163186
λ_2	-3.262424	1.017574	0.165531	0.049262
λ_3	-4.821636	0.906849	0.117557	0.019506
λ_4	-6.162036	0.838537	0.092305	0.007802

Table 2: Mean, standard deviation, skewness and kurtosis data for first four edge–scaled eigenvalues in the $\beta = 1$ Gaussian ensemble. Corollary 3.15 implies that rows 2 and 4 give corresponding data for the two largest eigenvalues in the $\beta = 4$ Gaussian ensemble. Compare to Table 1 in [35], keeping in mind that the discrepancy in the $\beta = 4$ data is caused by the different normalization in the definition of $F_4(s, 1)$.

F_1-Percentile	100×100			100×400		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.01	0.008	0.005	0.004	0.008	0.006	0.004
0.05	0.042	0.033	0.025	0.042	0.037	0.032
0.10	0.090	0.073	0.059	0.088	0.081	0.066
0.30	0.294	0.268	0.235	0.283	0.267	0.254
0.50	0.497	0.477	0.440	0.485	0.471	0.455
0.70	0.699	0.690	0.659	0.685	0.679	0.669
0.90	**0.902**	**0.891**	**0.901**	**0.898**	**0.894**	**0.884**
0.95	**0.951**	**0.948**	**0.950**	**0.947**	**0.950**	**0.941**
0.99	**0.992**	**0.991**	**0.991**	**0.989**	**0.991**	**0.989**

Table 3: Percentile comparison of F_1 vs. empirical distributions for 100×100 and 100×400 Wishart matrices with identity covariance.

References

[1] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Third Edition, Wiley-Interscience, 2003.

[2] G. E. Andrews, Askey R., and R. Ranjan, *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 2000.

[3] J. Baik and E. M. Rains, Algebraic aspects of increasing subsequences, *Duke Math. J.*, 109(1):1–65, 2001.

[4] M. Bilodeau and D. Brenner, *Theory of Multivariate Statistics*, Springer-Verlag, 1999.

[5] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, 1946.

[6] P. A. Clarkson and J. B. McLeod, A connection formula for the second Painlevé transcendent, *Arch. Rational Mech. Anal.*, 103(2):97–138, 1988.

[7] N. G. de Bruijn, On some multiple integrals involving determinants, *J. Indian Math. Soc.*, 19:133–151, 1955.

[8] P. Deift and D. Gioev, Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices, preprint, arXiv:math-ph/0507023.

[9] P. Deift and X. Zhou, Asymptotics for the Painlevé II equation, *Commun. Pure Appl. Math.*, 48:277–337, 1995.

[10] M. Dieng, Distribution functions for edge eigenvalues in orthogonal and symplectic ensembles: Painlevé representations, *Inter. Math. Res. Notices* 37:2263–2287, 2005.

[11] M. Dieng, Distribution functions for edge eigenvalues in orthogonal and symplectic ensembles: Painlevé representations, Ph.D. thesis, UC Davis, 2005.

[12] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, *J. Math. Phys.*, 43(11):5830–5847, 2002.

[13] A. Edelman and N. R. Rao, Random matrix theory, *Acta Numerica*, 14:233–297, 2005.

[14] N. El Karoui, On the largest eigenvalue of Wishart matrices with identity covariance when n, p and p/n tend to infinity, ArXiv:math.ST/0309355.

[15] P. L. Ferrari, Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues, *Commun. Math. Phys.* 252:77–109, 2004.

[16] P. J. Forrester, T. Nagao and G. Honner, Correlations for the orthogonal-unitary and symplectic-unitary transistions at the soft and hard edges, *Nucl. Phys. B*, 553:601–643, 1999.

[17] P. J. Forrester and E. M. Rains, Interrelationships between orthogonal, unitary and symplectic matrix ensembles, In P. Bleher, A. Its, and S. Levy, editors, *Random Matrix Models and their Applications*, volume 40 of *Math. Sci. Res. Inst. Publ.*, pages 171–207. Cambridge Univ. Press, Cambridge, 2001.

[18] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators, Vol. I*, volume 49 of *Operator Theory: Advances and Applications*, Birkhäuser, 1990.

[19] S. P. Hastings and J. B. McLeod, A boundary value problem associated with the second Painlevé transcendent and the Korteweg–de Vries equation, *Arch. Rational Mech. Anal.*, 73(1):31–51, 1980.

[20] A. T. James, Normal multivariate analysis and the orthogonal group, *Ann. Math. Statist.* 31:40–75, 1954.

[21] A. T. James, Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.* 35:475–501, 1964.

[22] I. M. Johnstone, On the distribution of the largest eigenvalue in principal component analysis, *Ann. Stats.*, 29(2):295–327, 2001.

[23] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford Univ. Press, 1995.

[24] M. L. Mehta, *Random Matrices, Revised and Enlarged Second Edition*, Academic Press, 1991.

[25] M. L. Mehta and F. J. Dyson, Statistical theory of the energy levels of complex systems. V., *J. Math. Phys.*, 4:713–719, 1963.

[26] R. J. Muirhead, *Aspects of Multivariate Statistical Theory*, John Wiley & Sons, 1982.

[27] K. Pearson, On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling, *Phil. Mag.*, V, 50:157, 1900.

[28] P. Persson, Numerical methods for random matrices, Course project for MIT 18.337, MIT, 2002.

[29] A. Soshnikov. A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices, *J. Stat. Phys.*, 108(5–6):1033–1056, 2002.

[30] R. Stanley. *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[31] C. A. Tracy and H. Widom. Level–spacing distributions and the Airy kernel, *Commun. Math. Physics*, 159:151–174, 1994.

[32] C. A. Tracy and H. Widom. Fredholm determinants, differential equations and matrix models, *Commun. Math. Physics*, 163:33–72, 1994.

[33] C. A. Tracy and H. Widom, On orthogonal and symplectic matrix ensembles, *Commun. Math. Physics*, 177:727–754, 1996.

[34] C. A. Tracy and H. Widom, Correlation functions, cluster functions, and spacing distributions for random matrices, *J. Stat. Phys.*, 92(5–6):809–835, 1998.

[35] C. A. Tracy and H. Widom, Airy kernel and Painlevé, In *Isomonodromic deformations and applications in physics*, volume 31 of *CRM Proceedings & Lecture Notes*, pages 85–98. Amer. Math. Soc., Providence, RI, 2002.

[36] C. A. Tracy and H. Widom, Matrix kernels for the Gaussian orthogonal and symplectic ensembles, *Ann. Inst. Fourier, Grenoble*, 55:2197–2207, 2005.

[37] H. Widom, On the relation between orthogonal, symplectic and unitary matrix ensembles, *J. Stat. Phys.* 94: 347–363, 1999.

[38] J. Wishart, The generalized product moment distribution in samples from a normal multivariate population, *Biometrika*, 20A:32–43, 1928.

[39] P. Zinn-Justin and J.-B. Zuber, On some integrals over $\mathbb{U}(N)$ unitary group and their large N limit, *J. Math. Phys. A: Math. Gen.*, 36:3173–3193, 2003.